

# MathQuest: Difference Equations

## Solving Homogeneous Systems of Difference Equations

1. If we are told that the general solution to a system of difference equations is

$$A_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix} = c_1 \cdot (0.9)^n \begin{bmatrix} 1 \\ \frac{7}{8} \end{bmatrix} + c_2(-0.5)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

then which is an equivalent form of the solution?

- (a)  $a_n = c_1(0.9)^n + \frac{7}{8}c_1(0.9)^n$  and  $b_n = -c_2(-0.5)^n + c_2(-0.5)^n$
  - (b)  $a_n = c_1(0.9)^n - c_2(-0.5)^n$  and  $b_n = \frac{7}{8}c_1(0.9)^n + c_2(-0.5)^n$
  - (c)  $a_n = c_1(0.9)^n - c_1(-0.5)^n$  and  $b_n = \frac{7}{8}c_2(0.9)^n + c_2(-0.5)^n$
  - (d) All of the above
  - (e) None of the above
2. The solution to a system of difference equations is

$$A_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix} = c_1 \cdot (0.9)^n \begin{bmatrix} 1 \\ \frac{7}{8} \end{bmatrix} + c_2(-0.5)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Which of the following is a true statement?

- (a) This system has an unstable equilibrium.
  - (b) In the long-run,  $b$  will hold  $7/8$  of the population.
  - (c) The equilibrium value of this system is  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .
  - (d) All of the above
  - (e) None of the above
3. If we wish to solve this system,

$$\begin{aligned} a_{n+1} &= a_n - 0.2a_n + 0.3b_n \\ b_{n+1} &= b_n - 0.3b_n \end{aligned}$$

which matrix do we need to find eigenvalues and eigenvectors for?

(a) 
$$\begin{bmatrix} 1 & -0.2 & 0.3 \\ 1 & -0.3 & 0 \end{bmatrix}$$

(b) 
$$\begin{bmatrix} 1 & -0.2 & 0.3 \\ 0 & 1 & -0.3 \end{bmatrix}$$

(c) 
$$\begin{bmatrix} 0.8 & 0.3 \\ 0.7 & 0 \end{bmatrix}$$

(d) 
$$\begin{bmatrix} 0.8 & 0.3 \\ 0 & 0.7 \end{bmatrix}$$

(e) None of the above

4. In solving the system

$$\begin{aligned} a_{n+1} &= a_n - 0.2a_n + 0.3b_n \\ b_{n+1} &= b_n - 0.3b_n \end{aligned}$$

we find that the eigenvalues of the coefficient matrix are 0.8 and 0.7 with corresponding eigenvectors of  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ . What is the solution to this system?

(a)  $A_n = c_1(0.8)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.7)^n \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

(b)  $A_n = c_1(0.8)^n \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2(0.7)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c)  $A_n = c_1(0.8) \begin{bmatrix} 1 \\ 0 \end{bmatrix}^n + c_2(0.7) \begin{bmatrix} -3 \\ 1 \end{bmatrix}^n$

(d)  $A_n = c_1(0.8) \begin{bmatrix} -3 \\ 1 \end{bmatrix}^n + c_2(0.7) \begin{bmatrix} 1 \\ 0 \end{bmatrix}^n$

(e) None of the above

5. The solution to a system of difference equations is  $A_n = c_1(0.8)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.7)^n \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ .

If  $A_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , what are  $c_1$  and  $c_2$ ?

(a)  $c_1 = 2$  and  $c_2 = 3$

(b)  $c_1 = 55/4$  and  $c_2 = 110/8$

- (c)  $c_1 = 11$  and  $c_2 = 3$
- (d)  $c_1 = -7$  and  $c_2 = 3$
- (e) None of the above

6. The following system of difference equations allows us to predict how the populations of two towns, A and B, change each year.

$$\begin{aligned} a_{n+1} &= a_n - 0.2a_n + 0.3b_n \\ b_{n+1} &= b_n - 0.3b_n \end{aligned}$$

The solution to this system is

$$A_n = c_1(0.8)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.7)^n \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Which of the following is a true statement?

- (a) This system has a stable equilibrium.
  - (b) In the long-run, both of these towns will be ghost towns.
  - (c) If there are initially 10,000 people in town B, then  $b_{10} = 282$  people.
  - (d) All of the above
  - (e) None of the above
7. If  $A_n = (2)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{1}{3}\right)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  is a solution to the system of difference equations  $A_{n+1} = RA_n$ , which of the following is also a solution?
- (a)  $(2^n) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  - (b)  $3 \cdot (2)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4 \cdot \left(\frac{1}{3}\right)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
  - (c)  $8 \cdot \left(\frac{1}{3}\right)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
  - (d) All of the above
  - (e) None of the above
8. **True or False** If either column of the coefficient matrix of a system of homogeneous difference equations sums to a value greater than one, then the system has an unstable equilibrium.
9. **True or False** When solving a system of two homogeneous difference equations, if one eigenvalue is greater than one and one is between 0 and 1, then one population will grow without bound while the other declines.