

## Linear Independence

1. **True or False** The following vectors are linearly independent:  $(1,0,0)$ ,  $(0,0,2)$ ,  $(3,0,4)$
- (a) True, and I am very confident
  - (b) True, but I am not very confident
  - (c) False, but I am not very confident
  - (d) False, and I am very confident

*Answer: (False).* We can write the third vector as a linear combination of the first two:  $v_3 = 3v_1 + 2v_2$ .

by Carroll College MathQuest

LA.00.06.002

WW JD MA289 W11: 12/27/27/**35**

CC KC MA334 S12: 42/8/0/**50**

HC AS MA339 F11: 11.11/16.67/27.78/**44.44**/0/0 ,

CC KC MA334 S13: 4/0/17/**79** time 2:00

2. Which set of vectors is linearly independent?

- (a)  $(2, 3), (8, 12)$
- (b)  $(1, 2, 3), (4, 5, 6), (7, 8, 9)$
- (c)  $(-3, 1, 0), (4, 5, 2), (1, 6, 2)$
- (d) None of these sets are linearly independent.
- (e) Exactly two of these sets are linearly independent.
- (f) All of these sets are linearly independent.

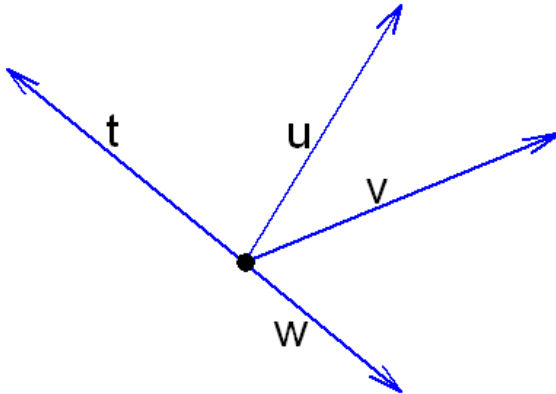
*Answer: (d).* In answer (a), the second vector is four times the first. In answer (b), we can get the third vector by subtracting the first one from twice the second vector. In answer (c), adding the first two vectors gives the third. This can be used immediately after defining linearly independent vectors, and students will likely use a variety of thought processes.

by Carroll College MathQuest

LA.00.06.010

CC KC MA334 S08: 25/50/0/**25**/0/0 time 5:00  
 CC KC MA334 S09: 4/19/0/**23**/50/4 time 4:00  
 CC KC MA334 S10: 10/30/5/**20**/35/0 time 5:00  
 KC MS MA224 F10: 5/35/0/**40**/20 time 4:45  
 CC KC MA334 S12: 0/69/4/**8**/19/0  
 CC KC MA334 S13: 0/12/0/**88**/0/0 time 4:00

3. Which subsets of the set of the vectors shown below are linearly dependent?



- (a)  $u, w$
- (b)  $t, w$
- (c)  $t, v$
- (d)  $t, u, v$
- (e) None of these sets are linearly dependent.
- (f) More than one of these sets is linearly dependent.

*Answer: (f).* Combinations (b) and (d) are linearly dependent sets. The vectors  $t$  and  $w$  are multiples of each other, so combination (b) is not a linearly independent set. Set (d) is not linearly independent because these vectors are in  $\mathbb{R}^2$  so a set of at most two vectors can be linearly independent. Thus vector  $t$  could be written as a linear combination of  $u$  and  $v$ , etc.

by Carroll College MathQuest

LA.00.06.015

CC KC MA334 S08: 0/44/6/0/0/**50** time 4:00  
 CC KC MA334 S09: 0/84/0/0/0/**16** time 3:00  
 CC KC MA334 S10: 0/29/0/0/0/**71** time 2:30  
 WW JD MA289 W11: 0/35/0/19/8/**38**  
 KC MS MA224 F10: 0/33/0/5/0/**62** time 2:00, voting mod-5, assuming no a votes  
 CC KC MA334 S12: 0/88/0/0/0/**12**

4. Suppose you wish to determine whether a set of vectors is linearly independent. You form a matrix with those vectors as the columns, and you calculate its reduced row

echelon form,  $R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . What do you decide?

- (a) These vectors are linearly independent.  
 (b) These vectors are not linearly independent.

*Answer: (b).* Since we only get three identity matrix columns out of four, these vectors are not linearly independent.

by Carroll College MathQuest

LA.00.06.020

CC KC MA334 S08: 41/59 time 4:30

CC KC MA334 S09: 10/90 time 3:00

CC KC MA334 S10: 5/95 time 1:30

SFCC GG MA220 Su10: 17/83 one clicker per pair

KC MS MA224 F09: 17/83 time 2:40

KC MS MA224 F10: 19/81 time 1:40

CC KC MA334 S12: 17/83

HC AS MA339 F11: 11.76/88.24

CC KC MA334 S13: 0/100

5. Suppose you wish to determine whether a set of vectors  $\{v_1, v_2, v_3, v_4\}$  is linearly independent. You form the matrix  $A = [v_1 v_2 v_3 v_4]$ , and you calculate its reduced row

echelon form,  $R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . You now decide to write  $v_4$  as a linear combination of  $v_1, v_2$ , and  $v_3$ . Which is a correct linear combination?

- (a)  $v_4 = v_1 + v_2$   
 (b)  $v_4 = -v_1 - 2v_3$   
 (c)  $v_4$  cannot be written as a linear combination of  $v_1, v_2$ , and  $v_3$ .  
 (d) We cannot determine the linear combination from this information.

*Answer: (a).* There are two ways to view this. Since we are trying to write the fourth column as a linear combination of the first three, students may view this matrix as the augmented matrix for the system  $Ax = v_4$ . Based on that view they set up the equations  $c_1 + 2c_3 = 1$  and  $c_2 + 3c_3 = 1$ , and letting  $c_3 = s$  they obtain the solution vector  $(1 - 2s, 1 - 3s, s)$ . Taking  $s = 0$  tells us that  $v_4 = v_1 + v_2$ . The other view is that the matrix is the coefficient matrix from the system  $Ax = 0$ , since the matrix arose while determining whether the set of vectors is linearly independent. In this case, coefficients for the linear combination must be of the form  $(-2s - t, -3s - t, s, t)^T$ . If we set  $t = 1$  so we can express the fourth vector as a linear combination of the others, we can set  $s = 0$  to find that  $-v_1 - v_2 + v_3 = 0$ , so  $v_3 = v_1 + v_2$ . Students who are more comfortable with augmented matrices than coefficient matrices may prefer the first view on this problem, but that likely will not help them answer the following two questions. A major point of this question is that once we determine that a set of vectors is not linearly independent, we do not have to start from scratch if we want to write one vector as a linear combination of the others.

by Carroll College MathQuest

LA.00.06.030

CC KC MA334 S09: **36**/36/0/27 time 4:00

CC KC MA334 S10: **24**/19/38/19 time 4:30

WW JD MA289 W11: **78**/15/7/0

KC MS MA224 F09: **52**/15/0/33 time 4:45

KC MS MA224 F10: **38**/24/5/33 time 3:20 "This led to the student conjecture that u,v,w are LI IFF u, u+v, u+v+w are LI"

CC KC MA334 S12: **30**/9/61/0

CC KC MA334 S13: **52**/8/32/8 time 4:00

6. Suppose you wish to determine whether a set of vectors  $\{v_1, v_2, v_3, v_4\}$  is linearly independent. You form the matrix  $A = [v_1 v_2 v_3 v_4]$ , and you calculate its reduced row

echelon form,  $R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . You now decide to write  $v_3$  as a linear combination of  $v_1, v_2$ , and  $v_4$ . Which is a correct linear combination?

(a)  $v_3 = (1/2)v_1 - (1/2)v_4$

(b)  $v_3 = (1/2)v_1 + (1/3)v_2$

(c)  $v_3 = 2v_1 + 3v_2$

(d)  $v_3 = -2v_1 - 3v_2$

(e)  $v_3$  cannot be written as a linear combination of  $v_1, v_2$ , and  $v_4$ .

(f) We cannot determine the linear combination from this information.

*Answer: (c).* Here we treat the matrix as the coefficient matrix from the system  $Ax = 0$ , since the matrix arose while determining whether the set of vectors is linearly independent. The coefficients for the linear combination must be of the form  $(-2s - t, -3s - t, s, t)^T$ . If we set  $s = 1$  so we can express the third vector as a linear combination of the others, we can set  $t = 0$  to find that  $-2v_1 - 3v_2 + v_3 = 0$ , so  $v_3 = 2v_1 + 3v_2$ . Students may also recognize that this is the same result as we would get by just observing this matrix and noting that the third column of  $R$  is equal to twice the first column of  $R$  plus three times the second, so row reduction doesn't change the coefficients of the linear combination. This question may be a bit harder for students than the previous one, because now the vector they wish to express as a linear combination is not the last column.

by Carroll College MathQuest

LA.00.06.040

CC KC MA334 S10: 0/24/**29**/14/10/24 time 6:00

WW JD MA289 W11: 0/0/**96**/4/0/0

CC KC MA334 S12: 8/12/**76**/4/0/0

CC KC MA334 S13: 4/28/**36**/28/4/0 time 6:00

7. Suppose you wish to determine whether a set of vectors  $\{v_1, v_2, v_3, v_4\}$  is linearly independent. You form the matrix  $A = [v_1 v_2 v_3 v_4]$ , and you calculate its reduced row echelon form,  $R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . You now decide to write  $v_2$  as a linear combination of  $v_1, v_3$ , and  $v_4$ . Which is a correct linear combination?
- (a)  $v_2 = 3v_3 + v_4$
  - (b)  $v_2 = -3v_3 - v_4$
  - (c)  $v_2 = v_4 - 3v_3$
  - (d)  $v_2 = -v_1 + v_4$
  - (e)  $v_2$  cannot be written as a linear combination of  $v_1, v_3$ , and  $v_4$ .
  - (f) We cannot determine the linear combination from this information.

*Answer: (d).* To get the second column of  $R$ , we can subtract the first column of  $R$  from the fourth. Formally, we treat the matrix as the coefficient matrix from the system  $Ax = 0$ , since the matrix arose while determining whether the set of vectors is linearly independent. The coefficients for the linear combination must be of the form  $(-2s - t, -3s - t, s, t)^T$ . We can take  $s = 0$  and  $t = -1$  to get  $v_1 + v_2 - v_4 = 0$ , so  $v_2 = -v_1 + v_4$ . The new point in this question is that even the vectors that reduced to identity matrix columns can sometimes be written as linear combinations of the other vectors.

by Carroll College MathQuest

LA.00.06.050

CC KC MA334 S10: 0/9/39/**52**/0/0 time 6:00

CC KC MA334 S13: 0/8/0/**92**/0/0 time 6:00

8. Are the vectors  $\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -14 \\ 13 \\ 7 \\ -19 \end{bmatrix} \right\}$  linearly independent?

- (a) Yes, they are linearly independent.
- (b) No, they are not linearly independent.

*Answer: (b).* If we call these vectors  $a, b, c,$  and  $d,$  then we see that  $d = 2a - 7b + 5c.$  More interesting will be the discussion of how the students thought about the question. We expect most students will use technology to put the corresponding matrix into reduced row echelon form. This question should lead into the more general question that follows.

by Carroll College MathQuest

LA.00.06.060

SFCC GG MA220 Su10: 0/**100** one clicker per pair

WW JD MA289 W11: 38/**62**

9. To determine whether a set of  $n$  vectors from  $\mathfrak{R}^n$  is independent, we can form a matrix  $A$  whose columns are the vectors in the set and then put that matrix in reduced row echelon form. If the vectors are linearly independent, what will we see in the reduced row echelon form?
- (a) A row of all zeros.
  - (b) A row that has all zeros except in the last position.
  - (c) A column of all zeros.
  - (d) An identity matrix.

*Answer: (d).* If the vectors are linearly independent, then  $Ax = 0$  will have the zero-vector as its only solution. This will be recognized by an identity matrix. Follow-up question: what will we see if the vectors are *not* linearly independent?

by Carroll College MathQuest

LA.00.06.070

HC AS MA330 F07: 0/0/9/**91**

SFCC GG MA220 Su10: 0/0/0/**100** one clicker per pair

10. To determine whether a set of fewer than  $n$  vectors from  $\mathfrak{R}^n$  is independent, we can form a matrix  $A$  whose columns are the vectors in the set and then put that matrix in reduced row echelon form. If the vectors are linearly independent, what will we see in the reduced row echelon form?
- (a) An identity submatrix with zeros below it.
  - (b) A row that has all zeros except in the last position.
  - (c) A column that is not an identity matrix column.
  - (d) A column of all zeros.

*Answer: (a).* If the vectors are linearly independent, then  $Ax = 0$  will have the zero-vector as its only solution. This will be recognized by an identity sub-matrix that has zeros below it. Follow-up question: what will we see if the vectors are *not* linearly independent?

by Carroll College MathQuest

LA.00.06.080

HC AS MA339 F07: 100/0/0/0

11. If the columns of  $A$  are not linearly independent, how many solutions are there to the system  $Ax = 0$ ?
- (a) 0
  - (b) 1
  - (c) infinite
  - (d) Not enough information is given.

*Answer: (c).* If the columns of  $A$  are not linearly independent, then one column can be written as a linear combination of the others, so there is a nonzero solution to  $Ax = 0$ . Since the zero-vector is also a solution, there are an infinite number of solutions to  $Ax = 0$ .

by Carroll College MathQuest

LA.00.06.090

SFCC GG MA220 Su10: 0/0/100/0 one clicker per pair

12. **True or False** A set of 4 vectors from  $\mathfrak{R}^3$  could be linearly independent.
- (a) True, and I am very confident
  - (b) True, but I am not very confident
  - (c) False, but I am not very confident
  - (d) False, and I am very confident

*Answer: False.* We are guaranteed, from the shape of the matrix with these vectors as columns, that one vector can be written as a linear combination of the other three.

by Carroll College MathQuest

LA.00.06.100

SFCC GG MA220 Su10: 0/0/0/100 one clicker per pair

13. **True or False** A set of 2 vectors from  $\mathfrak{R}^3$  must be linearly independent.
- (a) True, and I am very confident
  - (b) True, but I am not very confident
  - (c) False, but I am not very confident
  - (d) False, and I am very confident

*Answer: False.* One vector could just be a multiple of the other.

by Carroll College MathQuest

LA.00.06.110

SFCC GG MA220 Su10: 0/0/0/100 one clicker per pair

WW JD MA289 W11: 0/4/12/84

CC KC MA334 S13: 0/8/15/77 time 1:30

14. **True or False** A set of 3 vectors from  $\mathfrak{R}^3$  could be linearly independent.
- (a) True, and I am very confident
  - (b) True, but I am not very confident
  - (c) False, but I am not very confident
  - (d) False, and I am very confident

*Answer: True.* They *could* be independent, but they do not have to be.

by Carroll College MathQuest

LA.00.06.120

SFCC GG MA220 Su10: **100**/0/0/0 one clicker per pair

WW JD MA289 W11: **69**/15/4/12

CC KC MA334 S13: **88**/8/4/0 time 0:45

15. **True or False** A set of 5 vectors from  $\mathcal{R}^4$  could be linearly independent.

- (a) True, and I am very confident
- (b) True, but I am not very confident
- (c) False, but I am not very confident
- (d) False, and I am very confident

*Answer: False.* We are guaranteed, from the shape of the matrix with these vectors as columns, that one vector can be written as a linear combination of the other three.

by Carroll College MathQuest

LA.00.06.130

SFCC GG MA220 Su10: 0/0/0/**100** one clicker per pair

WW JD MA289 W11: 31/15/19/**35**

CC KC MA334 S13: 0/0/17/**83** time 1:30

16. Which statement is equivalent to saying that  $v_1, v_2$ , and  $v_3$  are linearly independent vectors?

- (a) The only solution to  $c_1v_1 + c_2v_2 + c_3v_3 = 0$  is  $c_1 = c_2 = c_3 = 0$ .
- (b)  $v_3$  cannot be written as a linear combination of  $v_1$  and  $v_2$ .
- (c) No vector is a multiple of any other.
- (d) Exactly two of (a), (b), and (c) are true.
- (e) All three statements are true.

*Answer: (a).* Answer (a) is equivalent because this is the definition of linear independence. Answer (b) does not imply independence because one of  $v_1$  or  $v_2$  could be a multiple of the other. You can correct answer (b) by changing it to read “no vector can be written as a linear combination of the other vectors.” Answer (c) does not imply independence because one vector could be a linear combination of the other two.

by Carroll College MathQuest

LA.00.06.140