

Eigenvalues and Eigenvectors

1. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(e) None of the above

Answer: (b). This is a straight-forward matrix multiplication problem, to help students discover the concept of eigenvectors.

CC KC MA232 S07: 0/100/0/0/0 time 1:00

CC HZ MA232 S08: 0/96/0/0/4 time 1:30

HHS JG MA232 S08: 0/100/0/0/0

by Carroll College MathQuest

LA.00.11.010

2. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$

(d) $\begin{bmatrix} 12 \\ 12 \end{bmatrix}$

(e) None of the above

(f) This matrix multiplication is impossible.

Answer: (c). We multiply a matrix to a power with an eigenvector, seeing the pattern emerge.

CC KC MA232 S07: 0/5/**95**/0 time 1:00

CC HZ MA232 S08: 0/0/**100**/0/0/0 time 1:05

HHS JG MA232 S08: 0/0/**100**/0/0/0

by Carroll College MathQuest

LA.00.11.020

3. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^4 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 27 \\ 27 \end{bmatrix}$

(b) $\begin{bmatrix} 81 \\ 81 \end{bmatrix}$

(c) $\begin{bmatrix} 243 \\ 243 \end{bmatrix}$

(d) $\begin{bmatrix} 729 \\ 729 \end{bmatrix}$

(e) None of the above

Answer: (b).

CC HZ MA232 S08: 0/**96**/0/4/0 time 0:45

HHS JG MA232 S08: 0/**100**/0/0/0

by Carroll College MathQuest

LA.00.11.030

4. For any integer n , what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 3n \\ 3n \end{bmatrix}$

(b) $3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $n^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $3^n \begin{bmatrix} n \\ n \end{bmatrix}$

(e) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}^n$

(f) More than one of the above

Answer: (b). We generalize the pattern, recognizing that a matrix to the n th times an eigenvector is simply the scalar eigenvalue to the n th times this eigenvector.

CC KC MA232 S07: 0/52/0/0/48/0 time 1:30

CC HZ MA232 S08: 8/72/4/0/12/4 time 0:55

HHS JG MA232 S08: 0/100/0/0/0/0

by Carroll College MathQuest

LA.00.11.040

5. Suppose A is an $n \times n$ matrix, c is a scalar, and x is an $n \times 1$ vector. If $Ax = cx$, what is A^2x ?

(a) $2cx$

(b) c^2x

(c) cx

(d) None of the above

Answer: (b). A generalization of the result.

by Carroll College MathQuest

LA.00.11.045

6. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$

(e) None of the above

Answer: (a). We look at another simple product to find another eigenvector.

CC KC MA232 S07: 100/0/0/0/0 time 1:30

CC HZ MA232 S08: **88**/8/4/0/0
HHS JG MA232 S08: **100**/0/0/0/0

by Carroll College MathQuest
LA.00.11.050

7. For any integer n , what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a) $(-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $(-1)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(c) $(-3)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$

(e) None of the above

(f) More than one of the above

Answer: (f). Both (a) and (d) are equivalent ways of generalizing this matrix to a power times an eigenvector.

CC KC MA232 S07: 67/19/0/5/0/**10** time 2:00

CC HZ MA232 S08: 62/19/0/12/8/**0** time 1:30

HHS JG MA232 S08: 70/20/0/0/10/**0**

by Carroll College MathQuest
LA.00.11.060

8. For any integer n , what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(a) $3^n \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $2^n \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(c) $6^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

Answer: (a). Here we see that a nonzero multiple of the first eigenvector that we found is also an eigenvector.

CC KC MA232 S07: **90**/0/5/0/0/5 time 2:00

CC HZ MA232 S08: **39**/4/35/0/23/0 time 2:05

HHS JG MA232 S08: **100**/0/0/0/0/0

by Carroll College MathQuest

LA.00.11.070

9. For any integer n , what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

(a) $3^n \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

(b) $(-1)^n \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

(c) $(-5)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $5 \begin{bmatrix} (-1)^n \\ (-1)^n \end{bmatrix}$

(e) None of the above

(f) More than one of the above

Answer: (b). Another demonstration that a nonzero multiple of an eigenvector is also an eigenvector.

CC KC MA232 S07: 5/**81**/0/0/14/0 time 2:00

HHS JG MA232 S08: 0/**100**/0/0/0/0

by Carroll College MathQuest

LA.00.11.080

10. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(a) $\begin{bmatrix} 3 \\ 15 \end{bmatrix}$

(b) $\begin{bmatrix} -1 \\ -5 \end{bmatrix}$

(c) $\begin{bmatrix} 11 \\ 7 \end{bmatrix}$

(d) $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$

(e) None of the above

Answer: (c). This is a straight-forward matrix/vector multiplication, but this time we see that it is not an eigenvector.

CC KC MA232 S07: 0/0/**100**/0/0 time 0:40

CC HZ MA232 S08: 0/0/**100**/0/0 time 0:35

HHS JG MA232 S08: 0/0/**100**/0/0

by Carroll College MathQuest

LA.00.11.090

11. For any integer n , what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(a) $11^n \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(b) $7^n \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 11^n \\ 7^n \end{bmatrix}$

(d) $\begin{bmatrix} 25 \\ 29 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

Answer: (e). We discover that there is no simple generalization of a matrix to a power times a non-eigenvector.

CC KC MA232 S07: 0/0/0/0/**100**/0 time 1:30

HHS JG MA232 S08: 0/0/0/0/**100**/0

by Carroll College MathQuest

LA.00.11.100

12. Write the vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$

(c) $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) None of the above

(e) More than one of the above

Answer: (c). We write this vector as a linear combination of the eigenvectors of the matrix. Answer (a) is a correct identity, so students may be lead to vote for (e), thinking that (a) and (c) are both correct. However, (a) is not what is asked for.

CC KC MA232 S07: 0/0/**67**/0/27 time 2:00

CC HZ MA232 S08: 3/0/**62**/3/31 time 1:50

HHS JG MA232 S08: 0/0/**30**/0/70

by Carroll College MathQuest

LA.00.11.110

13. For any integer n , what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(a) $-1 \times 3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \times (-2)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $3 \times (-1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-2) \times 3^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $3 \times 3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-2) \times (-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $3 \times 3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \times (-2)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

Answer: (c). Here we learn to calculate $A^n x$, when x is not an eigenvector, by first writing x as a linear combination of the eigenvectors of A .

CC KC MA232 S07: 0/0/**44**/25/13/6 time 5:00

CC HZ MA232 S08: 4/4/**68**/21/4/0 time 6:00

HHS JG MA232 S08: 0/0/**0**/70/30/0

by Carroll College MathQuest

LA.00.11.120

14. Which of the following is an eigenvector of the matrix $\begin{bmatrix} 2 & -1 \\ -4 & -1 \end{bmatrix}$?

(a) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

Answer: (c). We can see which of these is an eigenvector simply by multiplying the matrix by each and examining the result.

CC KC MA232 S07: 0/0/**82**/6/6/6 time 3:00

by Carroll College MathQuest

LA.00.11.130

15. Which of the following is an eigenvector of the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} -3 \\ -3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

Answer: (f). All four are eigenvectors. We can see that (a) and (b) are eigenvectors corresponding to the eigenvalues 4 and 1, respectively, by multiplying the matrix by each and examining the result. We further notice that (c) and (d) are multiples of (a) and (b), so they are also eigenvectors.

HC AS MA339 F07: 0/0/0/0/5/**95**

CC KC MA334 S09: 8/15/4/8/4/**61** time 4:30

by Carroll College MathQuest

LA.00.11.140

16. Suppose the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an eigenvalue 1 with associated eigenvector $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. What is $A^{50}x$?

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(b) $\begin{bmatrix} a^{50} & b^{50} \\ c^{50} & d^{50} \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2^{50} \\ 3^{50} \end{bmatrix}$

(e) Way too hard to compute.

Answer: (c). $A^{50}x = 1^{50}x = x$. This should be a quick check on students' understanding the computational significance of eigenvalues and eigenvectors.

HC AS MA339 F07: 5/0/85/5/5

CC KC MA334 S09: 0/15/23/42/0 time 2:30

by Carroll College MathQuest

LA.00.11.150

17. Vector x is an eigenvector of matrix A . If $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $Ax = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$, then what is the associated eigenvalue?

(a) 1

(b) 3

(c) 4

(d) Not enough information is given.

Answer: (c). $Ax = 4x$, so the associated eigenvalue is 4.

CC KC MA334 S09: 0/0/100/0 time 1:00

by Carroll College MathQuest

LA.00.11.155

18. Which of the following is an eigenvector of $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$? (You should be able to answer this by checking the vectors given, rather than by finding the eigenvectors of A from scratch.)

(a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

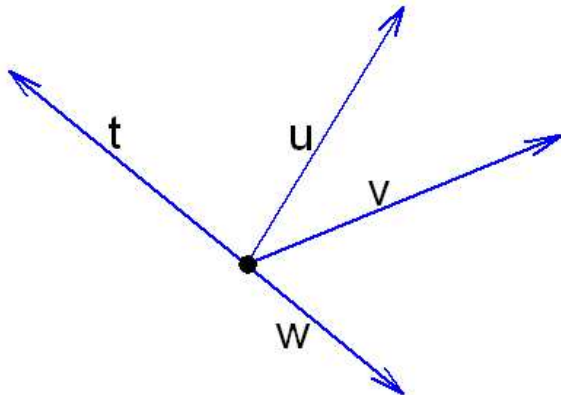
(d) None of the above

Answer: (c). We see that $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We can also learn from this that the associated eigenvalue is -2 , or that discussion can wait until after the next question.

by Carroll College MathQuest

LA.00.11.160

19. The vector t is an eigenvector of the matrix A . What could be the result of the product At ?



(a) $At = u$

(b) $At = v$

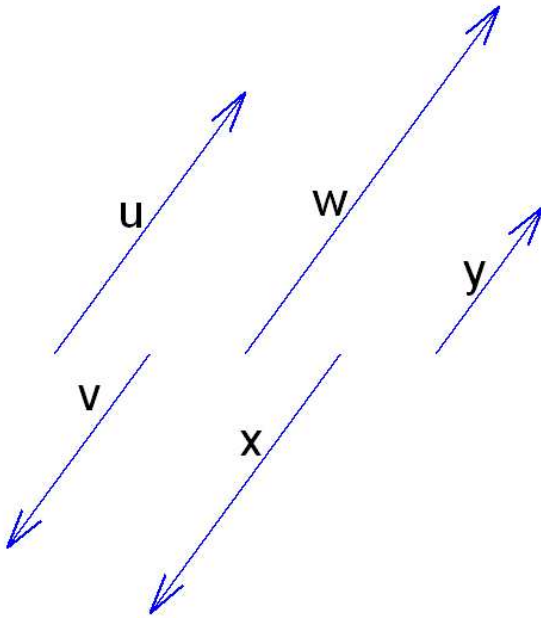
(c) $At = w$

(d) None of the above

Answer: (c). The vector w is parallel to the vector t : it can be produced by multiplying vector t by a scalar.

by Carroll College MathQuest
LA.00.11.165

20. The vector u is an eigenvector of the matrix A and $Au = v$, where the vectors u and v are shown below. What could be the result of the product Av ?



- (a) $Av = u$
- (b) $Av = v$
- (c) $Av = w$
- (d) $Av = x$
- (e) $Av = y$

Answer: (e). The vector v is shorter than u and points in the opposite direction which tells us that u must correspond to an eigenvalue between -1 and 0 . If we then take the product Av , we will multiply by this eigenvalue again, so the result will be shorter than v and point in the opposite direction, thus it must be y : $Av = A(\lambda u) = \lambda Au = \lambda^2 u = y$.

by Carroll College MathQuest
LA.00.11.167

21. $\begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$. What is the associated eigenvalue? (Think! Don't solve for all the eigenvalues and eigenvectors.)
- (a) $4/3$
 (b) 5
 (c) -2

Answer: (b). We see that $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 20/3 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$, so the associated eigenvalue is 5.

CC HZ MA232 S08: 11/78/11 time 5:40

CC KC MA334 S09: 20/80/0 time 3:00

by Carroll College MathQuest

LA.00.11.170

22. The matrix $A = \begin{bmatrix} -1 & 4 \\ 3 & 0 \end{bmatrix}$ has an eigenvalue 3 with associated eigenvector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Which of the following statements is true?
- (a) $Ax = 3x$
 (b) $Ay = 3y$
 (c) For any scalars c and d , $A(cx + dy) = 3(cx + dy)$
 (d) All of the above are true.
 (e) Only (a) and (b) are true.

Answer: (d). $y = 2x$, so y is also an eigenvector of A corresponding to the eigenvalue 3. Thus, both (a) and (b) are true. However, $A(cx + dy) = A(cx + 2dx) = A(c + 2d)x$. If we let $z = (c + 2d)x$, then z is also an eigenvector of A , so $Az = 3z = 3(cx + dy)$.

CC KC MA334 S09: 9/0/4/52/35 time 2:00

by Carroll College MathQuest

LA.00.11.180

23. The matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has an eigenvalue 2 with associated eigenvectors $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Which of the following statements is true?

- (a) $Ax = 2x$
- (b) $Ay = 2y$
- (c) For any scalars c and d , $A(cx + dy) = 2(cx + dy)$.
- (d) For any nonzero scalars c and d , $cx + dy$ is an eigenvector of A corresponding to the eigenvalue 2.
- (e) All of the above are true.
- (f) Only (a) and (b) are true.

Answer: (e). We know that (a) and (b) are true because x and y are both eigenvectors of A corresponding to the eigenvalue 2. Statements (c) and (d) are equivalent, with (c) easily demonstrated with the given matrix and vectors.

by Carroll College MathQuest

LA.00.11.190

24. **True or False** Any nonzero linear combination of two eigenvectors of a matrix A is an eigenvector of A .

Answer: False. This is not quite the correct generalization from the previous two questions. Ask the students to come up with the correct generalization: Any linear combination of two eigenvectors corresponding to the same eigenvalue of a matrix A is also an eigenvector of A corresponding to the same eigenvalue.

by Carroll College MathQuest

LA.00.11.200

25. If w is an eigenvector of A , how does the vector Aw compare geometrically to the vector w ?

- (a) Aw is a rotation of w .
- (b) Aw is a reflection of w in the x -axis.
- (c) Aw is a reflection of w in the y -axis.
- (d) Aw is parallel to w but may have a different length.

Answer: (d). Since $Aw = \lambda w$, we see that Aw is just a multiple of the eigenvector. Geometrically this means that only the length of the vector has changed (unless the eigenvalue is one, in which case the length is the same).

by Carroll College MathQuest

LA.00.11.210

26. What does it mean if 0 is an eigenvalue of a matrix A ?

- (a) The determinant of A is zero.
- (b) The columns of A are linearly dependent.
- (c) There are an infinite number of solutions to the system $Ax = 0$.
- (d) All of the above
- (e) None of the above

Answer: (d). Since λ is an eigenvalue of A when $\det(A - \lambda I) = 0$, a zero eigenvalue implies that $\det A = 0$. The other statements are equivalent to this one.

HC AS MA339 F07: 5/0/5/55/35

by Carroll College MathQuest

LA.00.11.220

27. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 4 & 2 \end{bmatrix}$ and note that all of the rows sum to six. Which of the following is true?

- (a) $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A .
- (b) 6 is an eigenvalue of A .
- (c) Both statements are true.
- (d) Neither statement is true.

Answer: (c). We can verify both statements by computing Aw . We will find that $Aw = 6w$, thus showing that w is an eigenvector of A corresponding to the eigenvalue 6. This question introduces students to the theorem that if the row sums of a real, square matrix are all equal, then that sum is an eigenvalue and the corresponding eigenvector has all ones. Ask students to explain why that theorem is true.

by Carroll College MathQuest

LA.00.11.230