MathQuest: Differential Equations

Linear Combinations and Independence of Functions

1. Which of the following expressions is a linear combination of the functions $f(t)$ and $g(t)$?

   (a) $2f(t) + 3g(t) + 4$
   (b) $f(t) - 2g(t) + t$
   (c) $2f(t)g(t) - 3f(t)$
   (d) $f(t) - g(t)$
   (e) All of the above
   (f) None of the above

2. **True or False** The function $h(t) = 4 + 3t$ is a linear combination of the functions $f(t) = (1 + t)^2$ and $g(t) = 2 - t - 2t^2$.

   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

3. **True or False** The function $h(t) = \sin(t + 2)$ is a linear combination of the functions $f(t) = \sin t$ and $g(t) = \cos t$.

   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

4. **True or False** $h(t) = t^2$ is a linear combination of $f(t) = (1 - t)^2$ and $g(t) = (1 + t)^2$.

   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

5. Let $y_1(t) = \sin(2t)$. For which of the following functions $y_2(t)$ will $\{y_1(t), y_2(t)\}$ be a linearly independent set?

   (a) $y_2(t) = \sin(t)\cos(t)$
   (b) $y_2(t) = 2\sin(2t)$
   (c) $y_2(t) = \cos(2t - \pi/2)$
   (d) $y_2(t) = \sin(-2t)$
   (e) All of the above
   (f) None of the above

6. Let $y_1(t) = e^{2t}$. For which of the following functions $y_2(t)$ will $\{y_1(t), y_2(t)\}$ be a linearly independent set?

   (a) $y_2(t) = e^{-2t}$
   (b) $y_2(t) = te^{2t}$
   (c) $y_2(t) = 1$
   (d) $y_2(t) = e^{3t}$
   (e) All of the above
   (f) None of the above

7. The functions $y_1(t)$ and $y_2(t)$ are linearly independent on the interval $a < t < b$ if

   (a) for some constant k, $y_1(t) = ky_2(t)$ for $a < t < b$.
   (b) there exists some $t_0 \in (a, b)$ and some constants c_1 and c_2 such that $c_1y_1(t_0) + c_2y_2(t_0) \neq 0$.
   (c) the equation $c_1y_1(t) + c_2y_2(t) = 0$ holds for all $t \in (a, b)$ only if $c_1 = c_2 = 0$.
   (d) the ratio $y_1(t)/y_2(t)$ is a constant function.
   (e) All of the above
   (f) None of the above

8. The functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $a < t < b$ if

   (a) there exist two constants c_1 and c_2 such that $c_1y_1(t)+c_2y_2(t) = 0$ for all $a < t < b$.
   (b) there exist two constants c_1 and c_2, not both 0, such that $c_1y_1(t) + c_2y_2(t) = 0$ for all $a < t < b$.
   (c) for each t in (a, b), there exists constants c_1 and c_2 such that $c_1y_1(t) + c_2y_2(t) = 0$.

(d) for some $a < t_0 < b$, the equation $c_1 y_1(t_0) + c_2 y_2(t_0) = 0$ can only be true if $c_1 = c_2 = 0$.

(e) All of the above

(f) None of the above

9. The functions $y_1(t)$ and $y_2(t)$ are both solutions of a certain second-order linear homogeneous differential equation with continuous coefficients for $a < t < b$. Which of the following statements are true?

(i) The general solution to the ODE is $y(t) = c_1 y_1(t) + c_2 y_2(t)$, $a < t < b$.

(ii) $y_1(t)$ and $y_2(t)$ must be linearly independent, since they both are solutions.

(iii) $y_1(t)$ and $y_2(t)$ may be linearly dependent, in which case we do not know enough information to write the general solution.

(iv) The Wronskian of $y_1(t)$ and $y_2(t)$ must be nonzero for these functions.

(a) Only (i) and (ii) are true.

(b) Only (i) is true.

(c) Only (ii) and (iv) are true.

(d) Only (iii) is true.

(e) None are true.

10. Can the functions $y_1(t) = t$ and $y_2(t) = t^2$ be a linearly independent pair of solutions for an ODE of the form

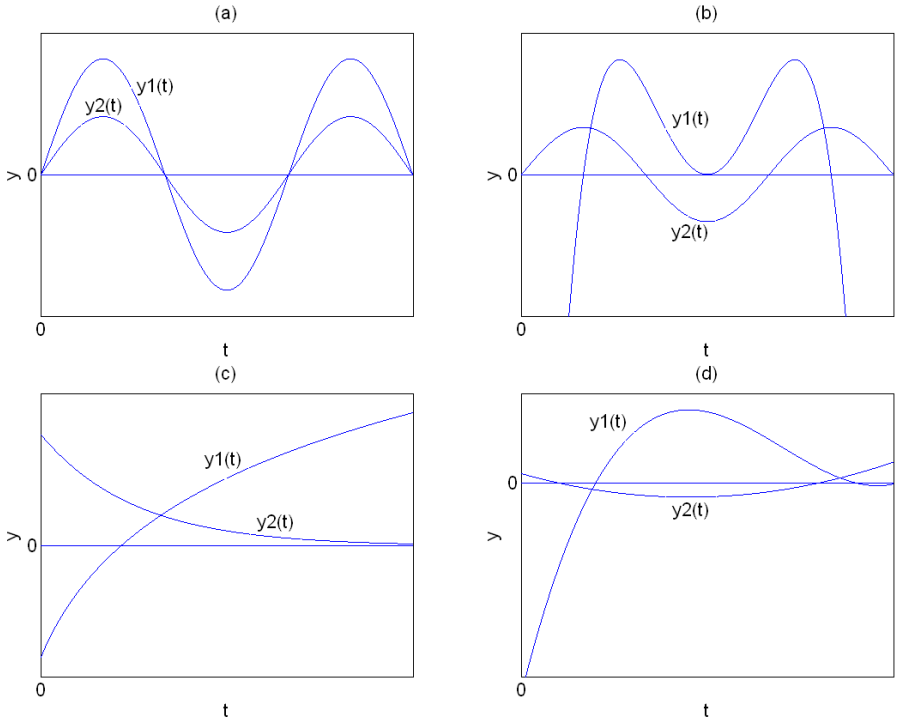
$$y'' + p(t)y' + q(t)y = 0 \qquad -1 \leq t \leq 1$$

where $p(t)$ and $q(t)$ are continuous functions?

(a) Yes

(b) No

11. Which pair of functions whose graphs are shown below could be linearly independent pairs of solutions to a second-order linear homogeneous differential equation?


(a)
y1(t)
y2(t)
y
0
0
t
(b)
y1(t)
y2(t)
y
0
0
t
(c)
y1(t)
y2(t)
y
0
0
t
(d)
y1(t)
y2(t)
0
y
0
t