

MathQuest: Differential Equations

Linear Combinations and Independence of Functions

- Which of the following expressions is a linear combination of the functions $f(t)$ and $g(t)$?
 - $2f(t) + 3g(t) + 4$
 - $f(t) - 2g(t) + t$
 - $2f(t)g(t) - 3f(t)$
 - $f(t) - g(t)$
 - All of the above
 - None of the above
- True or False** The function $h(t) = 4 + 3t$ is a linear combination of the functions $f(t) = (1 + t)^2$ and $g(t) = 2 - t - 2t^2$.
- True or False** The function $h(t) = \sin(t + 2)$ is a linear combination of the functions $f(t) = \sin t$ and $g(t) = \cos t$.
- True or False** $h(t) = t^2$ is a linear combination of $f(t) = (1 - t)^2$ and $g(t) = (1 + t)^2$.
- Let $y_1(t) = \sin(2t)$. For which of the following functions $y_2(t)$ will $\{y_1(t), y_2(t)\}$ be a linearly independent set?
 - $y_2(t) = \sin(t) \cos(t)$
 - $y_2(t) = 2 \sin(2t)$
 - $y_2(t) = \cos(2t - \pi/2)$
 - $y_2(t) = \sin(-2t)$
 - All of the above
 - None of the above
- Let $y_1(t) = e^{2t}$. For which of the following functions $y_2(t)$ will $\{y_1(t), y_2(t)\}$ be a linearly independent set?
 - $y_2(t) = e^{-2t}$

- (b) $y_2(t) = te^{2t}$
 - (c) $y_2(t) = 1$
 - (d) $y_2(t) = e^{3t}$
 - (e) All of the above
 - (f) None of the above
7. The functions $y_1(t)$ and $y_2(t)$ are linearly independent on the interval $a < t < b$ if
- (a) for some constant k , $y_1(t) = ky_2(t)$ for $a < t < b$.
 - (b) there exists some $t_0 \in (a, b)$ and some constants c_1 and c_2 such that $c_1y_1(t_0) + c_2y_2(t_0) \neq 0$.
 - (c) the equation $c_1y_1(t) + c_2y_2(t) = 0$ holds for all $t \in (a, b)$ only if $c_1 = c_2 = 0$.
 - (d) the ratio $y_1(t)/y_2(t)$ is a constant function.
 - (e) All of the above
 - (f) None of the above
8. The functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $a < t < b$ if
- (a) there exist two constants c_1 and c_2 such that $c_1y_1(t) + c_2y_2(t) = 0$ for all $a < t < b$.
 - (b) there exist two constants c_1 and c_2 , not both 0, such that $c_1y_1(t) + c_2y_2(t) = 0$ for all $a < t < b$.
 - (c) for each t in (a, b) , there exists constants c_1 and c_2 such that $c_1y_1(t) + c_2y_2(t) = 0$.
 - (d) for some $a < t_0 < b$, the equation $c_1y_1(t_0) + c_2y_2(t_0) = 0$ can only be true if $c_1 = c_2 = 0$.
 - (e) All of the above
 - (f) None of the above
9. The functions $y_1(t)$ and $y_2(t)$ are both solutions of a certain second-order linear homogeneous differential equation with continuous coefficients for $a < t < b$. Which of the following statements are true?
- (i) The general solution to the ODE is $y(t) = c_1y_1(t) + c_2y_2(t)$, $a < t < b$.
 - (ii) $y_1(t)$ and $y_2(t)$ must be linearly independent, since they both are solutions.
 - (iii) $y_1(t)$ and $y_2(t)$ may be linearly dependent, in which case we do not know enough information to write the general solution.
 - (iv) The Wronskian of $y_1(t)$ and $y_2(t)$ must be nonzero for these functions.

- (a) Only (i) and (ii) are true.
- (b) Only (i) is true.
- (c) Only (ii) and (iv) are true.
- (d) Only (iii) is true.
- (e) None are true.

10. Can the functions $y_1(t) = t$ and $y_2(t) = t^2$ be a linearly independent pair of solutions for an ODE of the form

$$y'' + p(t)y' + q(t)y = 0 \quad -1 \leq t \leq 1$$

where $p(t)$ and $q(t)$ are continuous functions?

- (a) Yes
- (b) No

11. Which pair of functions whose graphs are shown below could be linearly independent pairs of solutions to a second-order linear homogeneous differential equation?

