MathQuest: Differential Equations

Solutions to Linear Systems

1. Consider the linear system given by

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{Y}.$$

True or False: $\vec{Y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$ is a solution.

2. Consider the linear system $\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{Y}$ with solution $\vec{Y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$.

   True or False: The function $k \cdot \vec{Y}_1(t)$ formed by multiplying $\vec{Y}_1(t)$ by a constant k is also a solution to the linear system.

3. Consider the linear system $\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{Y}$. The functions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions to the linear system.

   True or False: The function $\vec{Y}_1(t) + \vec{Y}_2(t)$ formed by adding the two solutions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ is also a solution to the linear system.

4. **True or False:** The functions $\vec{Y}_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ and $\vec{Y}_2(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ are linearly independent.

5. **True or False:** The functions $\vec{Y}_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ and $\vec{Y}_2(t) = \begin{pmatrix} -2\sin(t) \\ -2\cos(t) \end{pmatrix}$ are linearly independent.

6. If we are told that the general solution to the linear homogeneous system $Y' = AY$ is $Y = c_1 e^{-4t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then an equivalent form of the solution is

   (a) $y_1 = -2c_1 e^{-4t} + c_2 e^{-4t}$ and $y_2 = 2c_1 e^{3t} + 3c_2 e^{3t}$

   (b) $y_1 = -2c_1 e^{-4t} + 2c_2 e^{3t}$ and $y_2 = c_1 e^{-4t} + 3c_2 e^{3t}$

   (c) $y_1 = -2c_1 e^{-4t} + c_1 e^{-4t}$ and $y_2 = 2c_2 e^{3t} + 3c_2 e^{3t}$

(d) $y_1 = -2c_1e^{-4t} + 2c_1e^{3t}$ and $y_2 = c_2e^{-4t} + 3c_2e^{3t}$

(e) All of the above

(f) None of the above

7. If $Y = e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a solution to the linear homogeneous system $Y' = AY$, which of the following is also a solution?

(a) $Y = 2e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $Y = 3e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4e^{-4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(c) $Y = 1/4e^{-4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(d) All of the above

(e) None of the above

8. The eigenvalues and eigenvectors for the coefficient matrix A in the linear homogeneous system $Y' = AY$ are $\lambda_1 = 4$ with $v_1 =< 1, 2 >$ and $\lambda_2 = -3$ with $v_2 =< -2, 1 >$. What is a form of the solution?

(a) $Y = c_1e^{-4t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $Y = c_1e^{4t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(c) $Y = c_1e^{4t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2e^{-3t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(d) $Y = c_1e^{-4t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

9. You have a linear, homogeneous, system of differential equations with constant coefficients whose coefficient matrix A has the eigensystem: eigenvalues -5 and -2 and eigenvectors $< -1, 2 >$ and $< -4, 5 >$, respectively. Then a general solution to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ is given by:

(a) $Y = \begin{bmatrix} -k_1e^{-5t} + 2k_2e^{-2t} \\ -4k_1e^{-5t} + 5k_2e^{-2t} \end{bmatrix}$

(b) $Y = \begin{bmatrix} -k_1e^{-2t} - 4k_2e^{-5t} \\ 2k_1e^{-2t} + 5k_2e^{-5t} \end{bmatrix}$

(c) $Y = \begin{bmatrix} -k_1e^{-5t} - 4k_2e^{-2t} \\ 2k_1e^{-5t} + 5k_2e^{-2t} \end{bmatrix}$

(d) $Y = \begin{bmatrix} -k_1e^{-2t} + 2k_2e^{-5t} \\ -4k_1e^{-2t} + 5k_2e^{-5t} \end{bmatrix}$

10. The eigenvalues and eigenvectors for the coefficient matrix A in the linear homogeneous system $Y' = AY$ are $\lambda_1 = 4$ with $v_1 =< 1, 2 >$ and $\lambda_2 = -3$ with $v_2 =< -2, 1 >$. In the long term, phase trajectories:

    (a) become parallel to the vector $v_2 =< -2, 1 >$.

    (b) tend towards positive infinity.

    (c) become parallel to the vector $v_1 =< 1, 2 >$.

    (d) tend towards 0.

    (e) None of the above

11. If the eigenvalues and eigenvectors for the coefficient matrix A in the linear homogeneous system $Y' = AY$ are $\lambda_1 = -4$ with $v_1 =< 1, 2 >$ and $\lambda_2 = 3$ with $v_2 =< 2, 3 >$, is $y_a = e^{-4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ a solution?

    (a) Yes, it is a solution.

    (b) No, it is not a solution because it does not contain λ_2.

    (c) No, it is not a solution because it is a vector.

    (d) No, it is not a solution because of a different reason.

12. The eigenvalues and eigenvectors for the coefficient matrix A in the linear homogeneous system $Y' = AY$ are $\lambda_1 = 4$ with $v_1 =< 1, 0 >$ and $\lambda_2 = 4$ with $v_2 =< 0, 1 >$. What is a form of the solution?

    (a) $Y = c_1e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2e^{4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

    (b) $Y = c_1e^{-4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2e^{-4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

    (c) $Y = c_1e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2te^{4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

    (d) $Y = c_1e^{-4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2e^{-4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

13. The system of differential equations $Y' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} Y$ has eigenvalue $\lambda = 2$ with multiplicity 2, and all eigenvectors are multiples of $v =< 1, -1 >$. Testing $Y = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we find that

   (a) Y is a solution.

   (b) Y is not a solution.

14. The system of differential equations $Y' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} Y$ has eigenvalue $\lambda = 2$ with multiplicity 2, and all eigenvectors are multiples of $v =< 1, -1 >$. One solution to this equation is $Y = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Testing $Y = te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we find that

   (a) $Y = te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is also a solution.

   (b) $Y = te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not a solution.

15. The eigenvalues and eigenvectors for the coefficient matrix A in the linear homogeneous system $Y' = AY$ are $\lambda = -4$ with multiplicity 2, and all eigenvectors are multiples of $v =< 1, -2 >$. What is the form of the general solution?

   (a) $Y = c_1 e^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 te^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

   (b) $Y = c_1 e^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

   (c) $Y = c_1 e^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \left(te^{-4t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + e^{-4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$