MathQuest: Differential Equations

Power Series Solutions

1. What is the solution to $\frac{df}{dx} = f$?

   (a) $f(x) = e^x$
   (b) $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$
   (c) Both of the above

2. Determine which of the following is a solution to the equation $y'' - xy = 0$ by taking derivatives and substituting them into the differential equation.

   (a) $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$
   (b) $y = x + \frac{1}{4!}x^4 + \frac{1}{7!}x^7 + \frac{1}{10!}x^{10} \cdots$
   (c) $y = 1 + \frac{1}{2\cdot3}x^3 + \frac{1}{2\cdot3\cdot5\cdot6}x^6 + \frac{1}{2\cdot3\cdot5\cdot6\cdot8\cdot9}x^9 + \cdots$
   (d) $y = 1 + \frac{1}{2}x^2 + \frac{1}{2\cdot4}x^4 + \frac{1}{2\cdot4\cdot6}x^6 + \cdots$
   (e) None of the above

3. Write the series $\sum_{n=2}^{\infty}(n+1)a_n x^n$ as an equivalent series whose first term corresponds to $n = 0$ rather than $n = 2$.

   (a) $\sum_{n=0}^{\infty}(n+1)a_n x^n$
   (b) $\sum_{n=0}^{\infty}(n+3)a_{n+2} x^{n+2}$
   (c) $\sum_{n=0}^{\infty}(n-1)a_{n-2} x^{n-2}$
   (d) $\sum_{n=0}^{\infty}(n-1)a_{n+2} x^{n+2}$

4. What is the second derivative of the function defined by the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$?

   (a) $y''(x) = \sum_{n=0}^{\infty} n(n-1)a_{n-2} x^{n-2}$
   (b) $y''(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1} x^n$
   (c) $y''(x) = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2} x^n$
   (d) $y''(x) = \sum_{n=0}^{\infty}(n-2)(n-3)a_{n-2} x^{n-4}$

5. Find a power series in the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ that is a solution for the differential equation $y' = 3y$, by taking the derivative of this series, substituting both the series and its derivative into the differential equation, and simplifying the result.

   (a) $a_{n+1} = \frac{3}{n+1} a_n$
   (b) $a_{n+1} = \frac{-3}{n+1} a_n$
   (c) $a_{n+1} = \frac{3}{n-1} a_n$
   (d) $a_{n+1} = \frac{-3}{n-1} a_n$

6. Which of the following power series is a solution to the differential equation $y' = -2y$?

   (a) $y(x) = 2 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$
   (b) $y(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$
   (c) $y(x) = -2 - x - \frac{2}{3}x^2 - \frac{1}{2}x^3 + \cdots$
   (d) $y(x) = 2 - 4x + 4x^2 - \frac{8}{3}x^3 + \frac{4}{3}x^4 + \cdots$
   (e) $y(x) = -1 - 2x - 2x^2 - \frac{4}{3}x^3 - \frac{2}{3}x^4 + \cdots$

7. Find a power series in the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ that is a solution for the differential equation $y''+4y = 0$, by taking the second derivative of this series, substituting both the series and its second derivative into the differential equation, and finding a difference equation for a_n.

   (a) $a_{n+2} = \frac{-4}{(n+1)(n+2)} a_n$
   (b) $a_{n+2} = \frac{-4}{(n-1)(n-2)} a_n$
   (c) $a_{n+2} = \frac{-4}{n(n+1)} a_n$
   (d) $a_{n+2} = \frac{-4}{n(n-1)} a_n$

8. Which of the following is a solution to $(x-2)y'' - y = 0$?

   (a) $x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \cdots$
   (b) $(x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{12}(x-2)^3 + \frac{1}{144}(x-2)^4 + \cdots$
   (c) $(x+2) + \frac{1}{2}(x+2)^2 + \frac{1}{12}(x+2)^3 + \frac{1}{144}(x+2)^4 + \cdots$
   (d) $-144 \cdot (x-2) - 72(x-2)^2 - 12(x-2)^3 - (x-2)^4 + \cdots$
   (e) None of the above
   (f) More than one of the above

9. Find a power series in the form $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ that is a solution for the differential equation $y'' - xy = 0$. Hint: After subsituting in the series for y and y'' it may be useful to write $x = 1 + (x-1)$.

(a) $a_{n+2} = \frac{a_n + a_{n-1}}{(n+1)(n+2)}$

(b) $a_{n+2} = \frac{a_n + a_{n-1}}{(n)(n+1)}$

(c) $a_{n+2} = \frac{a_n + a_{n+1}}{(n+1)(n+2)}$

(d) $a_{n+2} = \frac{a_n + a_{n+1}}{(n)(n+1)}$