

## MathQuest: Differential Equations

### Power Series Solutions

1. What is the solution to  $\frac{df}{dx} = f$ ?
  - (a)  $f(x) = e^x$
  - (b)  $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$
  - (c) Both of the above
2. Determine which of the following is a solution to the equation  $y'' - xy = 0$  by taking derivatives and substituting them into the differential equation.
  - (a)  $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$
  - (b)  $y = x + \frac{1}{4!}x^4 + \frac{1}{7!}x^7 + \frac{1}{10!}x^{10} \dots$
  - (c)  $y = 1 + \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}x^6 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}x^9 + \dots$
  - (d)  $y = 1 + \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 + \frac{1}{2 \cdot 4 \cdot 6}x^6 + \dots$
  - (e) None of the above
3. Write the series  $\sum_{n=2}^{\infty} (n+1)a_n x^n$  as an equivalent series whose first term corresponds to  $n = 0$  rather than  $n = 2$ .
  - (a)  $\sum_{n=0}^{\infty} (n+1)a_n x^n$
  - (b)  $\sum_{n=0}^{\infty} (n+3)a_{n+2} x^{n+2}$
  - (c)  $\sum_{n=0}^{\infty} (n-1)a_{n-2} x^{n-2}$
  - (d)  $\sum_{n=0}^{\infty} (n-1)a_{n+2} x^{n+2}$
4. What is the second derivative of the function defined by the power series  $y(x) = \sum_{n=0}^{\infty} a_n x^n$ ?
  - (a)  $y''(x) = \sum_{n=0}^{\infty} n(n-1)a_{n-2} x^{n-2}$
  - (b)  $y''(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$
  - (c)  $y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n$
  - (d)  $y''(x) = \sum_{n=0}^{\infty} (n-2)(n-3)a_{n-2} x^{n-4}$

5. Find a power series in the form  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  that is a solution for the differential equation  $y' = 3y$ , by taking the derivative of this series, substituting both the series and its derivative into the differential equation, and simplifying the result.

(a)  $a_{n+1} = \frac{3}{n+1}a_n$

(b)  $a_{n+1} = \frac{-3}{n+1}a_n$

(c)  $a_{n+1} = \frac{3}{n-1}a_n$

(d)  $a_{n+1} = \frac{-3}{n-1}a_n$

6. Which of the following power series is a solution to the differential equation  $y' = -2y$ ?

(a)  $y(x) = 2 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots$

(b)  $y(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots$

(c)  $y(x) = -2 - x - \frac{2}{3}x^2 - \frac{1}{2}x^3 + \dots$

(d)  $y(x) = 2 - 4x + 4x^2 - \frac{8}{3}x^3 + \frac{4}{3}x^4 + \dots$

(e)  $y(x) = -1 - 2x - 2x^2 - \frac{4}{3}x^3 - \frac{2}{3}x^4 + \dots$

7. Find a power series in the form  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  that is a solution for the differential equation  $y'' + 4y = 0$ , by taking the second derivative of this series, substituting both the series and its second derivative into the differential equation, and finding a difference equation for  $a_n$ .

(a)  $a_{n+2} = \frac{-4}{(n+1)(n+2)}a_n$

(b)  $a_{n+2} = \frac{-4}{(n-1)(n-2)}a_n$

(c)  $a_{n+2} = \frac{-4}{n(n+1)}a_n$

(d)  $a_{n+2} = \frac{-4}{n(n-1)}a_n$

8. Which of the following is a solution to  $(x - 2)y'' - y = 0$ ?

(a)  $x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \dots$

(b)  $(x - 2) + \frac{1}{2}(x - 2)^2 + \frac{1}{12}(x - 2)^3 + \frac{1}{144}(x - 2)^4 + \dots$

(c)  $(x + 2) + \frac{1}{2}(x + 2)^2 + \frac{1}{12}(x + 2)^3 + \frac{1}{144}(x + 2)^4 + \dots$

(d)  $-144 \cdot (x - 2) - 72(x - 2)^2 - 12(x - 2)^3 - (x - 2)^4 + \dots$

(e) None of the above

(f) More than one of the above

9. Find a power series in the form  $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$  that is a solution for the differential equation  $y'' - xy = 0$ . Hint: After substituting in the series for  $y$  and  $y''$  it may be useful to write  $x = 1 + (x-1)$ .

(a)  $a_{n+2} = \frac{a_n + a_{n-1}}{(n+1)(n+2)}$

(b)  $a_{n+2} = \frac{a_n + a_{n-1}}{(n)(n+1)}$

(c)  $a_{n+2} = \frac{a_n + a_{n+1}}{(n+1)(n+2)}$

(d)  $a_{n+2} = \frac{a_n + a_{n+1}}{(n)(n+1)}$