MathQuest: Differential Equations

Equilibria and Stability

1. The differential equation $\frac{dy}{dt} = (t-3)(y-2)$ has equilibrium values of

   (a) $y = 2$ only

   (b) $t = 3$ only

   (c) $y = 2$ and $t = 3$

   (d) No equilibrium values

   Answer: (a). An equilibrium value is a value of the function y, that makes the derivative $\frac{dy}{dt} = 0$ for all t, so if we choose this as our initial condition, then the function will always remain at this value.

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2. Suppose that 3 is an equilibrium value of a differential equation. This means that

   (a) the values will approach 3.

   (b) if the initial value is below 3, the values will decrease.

   (c) if the initial value is 3, then all of the values will be 3.

   (d) all of the above.

   Answer: (c). If we choose an equilibrium value as our initial condition, this means that the derivative will be zero, so we will stay at this value.

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3. We know that a given differential equation is in the form $y' = f(y)$, where f is a differentiable function of y. Suppose that $f(5) = 2$ and $f(-1) = -6$.

   (a) y must have an equilibrium value between $y = 5$ and $y = -1$.

   (b) y must have an equilibrium value between $y = 2$ and $y = -6$.

   (c) This does not necessarily indicate that any equilibrium value exists.

   Answer: (a). The function $f(y)$ gives us the derivative of y and if it is continuous, this means that at some point between $y = 5$ and $y = -1$ there must be a value where $y' = f(y) = 0$, and that makes it an equilibrium value.

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   AS DH 3335 010 S15: **50**/20/30 time 4:10

4. We know that a given differential equation is in the form $y' = f(y)$, where f is a differentiable function of y. Suppose that $f(10) = 0$, $f(9) = 3$, and $f(11) = -3$.

   (a) This means that $y = 10$ is a stable equilibrium.

(b) $y = 10$ is an equilibrium, but it might not be stable.

(c) This does not tell us for certain that $y = 10$ is an equilibrium.

Answer: (b). $f(9) = 3$ means that when $y = 3$ the function is increasing, and $f(11) = -3$ means that when $y = 11$ the function is decreasing. This might lead us to believe that solutions must go towards the equilibrium at $y = 10$, making it a stable equilibrium. However since we don't know the function, we can't be sure this is true. $y = 10$ might be unstable, with two stable equilibria located at $y = 9.5$ and $y = 10.5$.

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5. We know that a given differential equation is in the form $y' = f(y)$, where f is a differentiable function of y. Suppose that $f(6) = 0$, $f(14) = 0$, and $y(10) = 10$.

   (a) This means that $y(0)$ must have been between 6 and 14.

   (b) This means that $y(20) = 0$ is impossible.

   (c) This means that $y(20) = 20$ is impossible.

   (d) All of the above.

   (e) None of the above.

   Answer: (d). A solution trajectory can never cross an equilibrium solution trajectory, so if $y(10) = 10$, this means that all past and future values of y are confined between the equilibria at $y = 6$ and $y = 14$.

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   CC HZ MA232 S10: 3/9/6/**29**/54 time 5:30 Review
   CC HZ MA232 S11: 12/0/6/**6**/71 time 3:00

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AS DH MA3335 010 S12: 8/15/8/**8**/62 time 4:00
CC HZ MA232 S13: 13/4/0/**30**/52 time 3:45

6. We know that a given differential equation is in the form $y' = f(y)$, where f is a differentiable function of y. Suppose that $f(2) = 3$ and that $y(0) = 0$. Which of the following is impossible?

   (a) $y(10) = 6$

   (b) $y(10) = -6$

   (c) $y(-10) = 6$

   (d) $y(-10) = -6$

   (e) All of these are possible

   Answer: (c). When $y = 2$, we know that $y' = 3$ so the function $y(t)$ is increasing. If $y(-10) = 6$, then for $t > -10$, y must always be larger than $y = 2$: If it went down through $y = 2$, then y' would have to be negative.

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   CC HZ MA232 S13: 4/29/**0**/0/67 time 2:15

7. We know that a given differential equation is in the form $y' = f(y)$, where f is a differentiable function of y. Suppose that $f(5) = -2$, $f(10) = 4$, and that $y(10) = 3$.

   (a) $y(0)$ must be below 5.

   (b) $y(20)$ must be below 5.

   (c) $y(5)$ could be above 10.

   (d) $y(15)$ must be less than 3.

   Answer: (b). $y(20)$ must be below 5, because we know that y is decreasing when $y = 5$. Because $y(10) = 3$ this means that in the future, y could never rise above 5. (a) is not true because we know that the function is decreasing when $y = 5$. Thus y could have started out above 5, and then decreased to 3 when $t = 10$. (c) is not true because we know that the function is increasing when $y = 10$, so if $y(5)$ was greater then 10, then y would always have to be greater than 10 for $y > 5$. (d) is not true because we don't know what $f(3)$ is. If $f(3)$ was positive, then $y(15)$ could be larger than 3. We can only say for certain that it must be below $y = 5$.

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8. A differential equation has a stable equilibrium value of $T = 6$. Which of the following functions is definitely not a solution?

   (a) $T(t) = 5e^{-3t} + 6$

   (b) $T(t) = -4e^{-2t} + 6$

   (c) $T(t) = 4e^{2t} + 10$

   (d) They could all be solutions

   Answer: (d). Both a and b asymptotically approach 6. c diverges from 10, but just because 6 is a stable equilibrium doesn't mean that the equation does not also have an unstable equilibrium at 10.

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   HHS JG MA232 S08: 33/0/58/**8**

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9. Consider the differential equation $\frac{df}{dx} = \sin(f)$

   (a) $f = 0$ is a stable equilibrium.

   (b) $f = 0$ is an unstable equilibrium.

   (c) $f = 0$ is not an equilibrium.

   Answer: (b). $f = 0$ is an equilibrium, because it makes the derivative $\frac{df}{dx} = 0$. This equilibrium is unstable because initial conditions slightly above this equilibrium will give us positive derivatives, pushing the solution trajectory away from equilibrium, while conversely initial conditions slightly below equilibrium give us negative derivatives, also pushing solutions away from equiliibrium.

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   CC KC MA232 S07: 24/**48**/29 time 2:00
   CC HZ MA232 S08: 38/**62**/0 time 2:10 Review
   AU CS MA244 S08: 10/**74**/16
   CC JS MA232 S09: 40/**50**/10

CC HZ MA232 S10: 20/**67**/13 time 3:00
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10. Consider the differential equation $\frac{df}{dx} = af + b$, where a and b are positive parameters. If we increase b, what will happen to the equilibrium value?

    (a) it increases

    (b) is decreases

    (c) it stays the same

    (d) not enough information is given

    Answer: (b). The equilibrium value of this equation is $-b/a$, so if b becomes larger, this negative quantity decreases.

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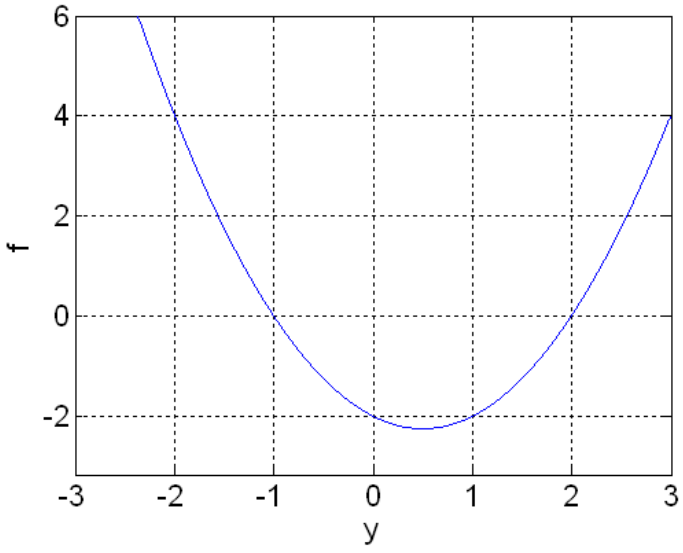
11. Suppose that $\frac{dy}{dt} = f(y)$, which is plotted below. What are the equilibrium values of the system?



    (a) $y = \frac{1}{2}$ is the only equilibrium.

(b) $y = -1$ and $y = 2$ are both equilibria.

(c) Not enough information is given.

Answer: (b). According to the graph $y' = f(y) = 0$ when $y = -1$ and $y = 2$, so these are equilibria.

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AU CS MA244 S08: 0/**92**/8
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CC HZ MA232 S10: 17/**67**/17 time 2:30
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AS DH MA3335 010 S12: 8/**0**/92 time 3:00
CC HZ MA232 S13: 13/**56**/26 time 3:00
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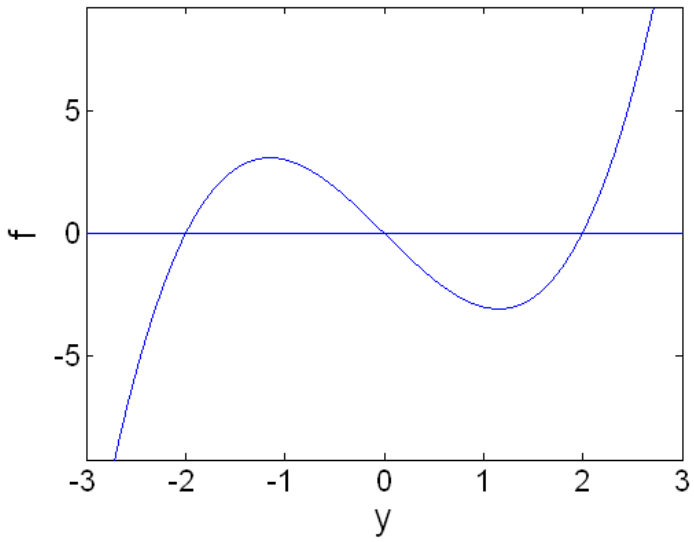
12. Suppose that $\frac{dy}{dt} = f(y)$, which is plotted below. What can we say about the equilibria of this system?



(a) $y = 0$ is stable, $y = \pm 2$ are unstable.

(b) $y = 0$ is unstable, $y = \pm 2$ are stable.

(c) $y = -2, 0$ are stable, $y = 2$ is unstable.

(d) $y = -2$ is unstable, $y = 0, 2$ are unstable

(e) None of the above

Answer: (a). Equilibria are points where $y' = f(y) = 0$ so the equation has equilibria at $y = 0, \pm 2$. A stable equilibrium is a point where f changes sign from positive to negative, so $y = 0$ is stable, while conversely $y = \pm 2$ are unstable.

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HHS JG MA232 S08: **36**/45/9/0/9
AU CS MA244 S08: **63**/13/0/0/25
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AS DH MA3335 010 S12: **100**/0/0/0/0 time 3:00

13. **True or False** A differential equation could have infinitely many equilibria.

    (a) True, and I am very confident

    (b) True, but I am not very confident

    (c) False, but I am not very confident

    (d) False, and I am very confident

    Answer: (True). For example if $\frac{dy}{dx} = \sin y$ then equilibria exists at $y = n\pi$ for any integer n.

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14. **True or False** A differential equation could have infinitely many equilibria over a finite range.

    (a) True, and I am very confident

    (b) True, but I am not very confident

    (c) False, but I am not very confident

    (d) False, and I am very confident

Answer: (True). For example if $\frac{dy}{dx} = 0$ then all values of y are equilibria.

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15. Consider the differential equation $\frac{df}{dx} = af + b$, where a and b are non-negative parameters. This equation would have no equilibrium if

    (a) $a = 0$

    (b) $b = 0$

    (c) $a = 1$

    (d) More than one of the above

    Answer: (a). If $a = 0$ the differential equation is just $\frac{df}{dx} = b$ where b is a positive number. This will result in an increasing function, so there can be no equilibrium.

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16. What is the equilibrium value of $\frac{dg}{dz} = -\frac{1}{2}g + 3e^z$?

    (a) This system is at equilibrium when $g = 6e^z$.

    (b) This system is at equlibrium when $z = \ln\left(\frac{g}{6}\right)$.

    (c) Both a and b are true.

    (d) This equation has no equlibrium.

    Answer: (d). An equilibrium would be a value of the function g, that would make $g' = 0$ for all values of z. In this case, there is no such value.

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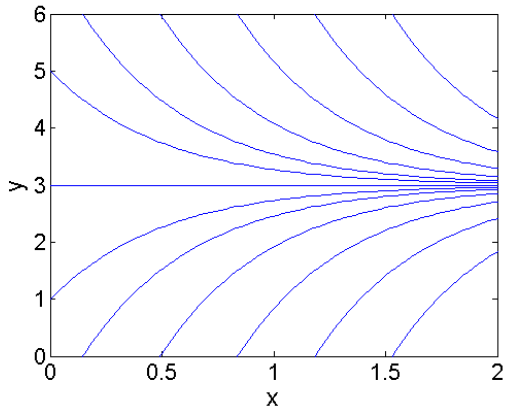
17. The figure below plots several functions which all solve the differential equation $y' = ay + b$. What could be the values of a and b?



(a) $a = 1$, $b = 3$

(b) $a = 2$, $b = -6$

(c) $a = -1$, $b = -3$

(d) $a = -2$, $b = 6$

(e) $b = 3$ but a is not easy to tell

Answer: (d). The plot shows the values leveling off at $y = 3$, so the system has a stable equilibrium at $y = 3$. Because the equilibrium is stable, this means that $a < 0$. To be an equilibrium value we need $-\frac{b}{a} = 3$.

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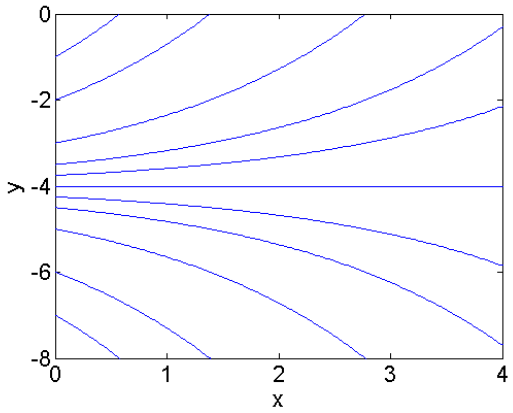
CC HZ MA232 S13: 12/21/8/**38**/21 time 2:45 Review
AS DH 3335 010 S13: 17/7/13/**0**/63 time 3:50

18. The figure below plots several functions which all solve the differential equation $\frac{dy}{dx} = ay + b$. What could be the values of a and b?



(a) $a = 0.5$, $b = 2$

(b) $a = 0.5$, $b = -2$

(c) $a = -0.5$, $b = 2$

(d) $a = -0.5$, $b = -2$

(e) None of the above are possible.

Answer: (a). The plot shows an unstable equilibrium at $y = -4$. This means that a must be positive and $y_{eq} = -\frac{b}{a} = -4$, which is satisfied if $a = 0.5$ and $b = 2$.

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