Second Order Differential Equations: Forcing

1. The three functions plotted below are all solutions of $y'' + ay' + 4y = \sin x$. Is $a$ positive or negative?

(a) $a$ is positive.
(b) $a$ is negative.
(c) $a = 0$.
(d) Not enough information is given.

Answer: (a). $a$ is positive. All three functions start with different initial conditions but converge to the same behavior. This means that the initial conditions die out, leaving only motion controlled by the driving function $\sin(x)$. The characteristic equation must have negative real roots, so there must be positive damping in the equation.

by Carroll College MathQuest
DEQ.10.16.010
CC KC MA334 S08: 83/0/0/17 time 3:00 ($a = 0$ was not an option)
CC KC MA334 S09: 21/12/25/42 time 3:00
CC KC MA334 S13: 67/8/0/25 time 5:00

2. If we conjecture the function $y(x) = C_1 \sin 2x + C_2 \cos 2x + C_3$ as a solution to the differential equation $y'' + 4y = 8$, which of the constants is determined by the differential equation?

(a) $C_1$
(b) $C_2$
(c) $C_3$
(d) None of them are determined.

*Answer:* (c). $C_3$ is determined. If we take two derivatives of this function we get $-4C_1 \sin 2x - 4C_2 \cos 2x$. If we put this into the differential equation, we find that all the sine and cosine terms cancel out, allowing us to solve for $C_3 = 2$.

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3. What will the solutions of $y'' + ay' + by = c$ look like if $b$ is negative and $a$ is positive.

(a) Solutions will oscillate at first and level out at a constant.
(b) Solutions will grow exponentially.
(c) Solutions will oscillate forever.
(d) Solutions will decay exponentially.

*Answer:* (b). Solutions will grow exponentially. If $b$ was positive, our solutions would contain sines and cosines, but because $b$ is negative, our solutions are growing and decaying exponentials. Except for a very special set of initial conditions, most solutions will contain both a growing and a decaying part, with the growing part dominating the long term behavior.

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4. The functions plotted below are solutions to which of the following differential equations?
(a) \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4y = 3 - 3x \)
(b) \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4y = 3e^{2x} \)
(c) \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4y = \sin \frac{2\pi}{9} x \)
(d) \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 4y = 2x^2 + 3x - 4 \)
(e) None of the above

**Answer:** (a). The left side of the differential equation can be thought to represent a damped harmonic oscillator, and the terms on the right side are a forcing function. Over time, solution trajectories will converge to a solution with the form of the forcing function. These solutions are converging to a linear function, so they must be responding to a linear forcing function. This is question is intended to be given before we have learned the precise method of solving this type of equation.

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AS DH MA3335 010 S12: 0/0/100/0/0 time 2:30
AS DH 3335 010 S13: 13/21/67/0/0 time 3:10
CC KC MA334 S13: 82/0/4/0/14 time 3:00
AS DH 3335 010 S14: 8/25/21/25/21 time 3:00
AS DH 3335 010 S15: 54/0/15/0/31 time 5:00

5. The general solution to \( f'' + 7f' + 12f = 0 \) is \( f(t) = C_1e^{-3t} + C_2e^{-4t} \). What should we conjecture as a particular solution to \( f'' + 7f' + 12f = 5e^{-2t} \)?

(a) \( f(t) = Ce^{-4t} \)
(b) \( f(t) = C e^{-3t} \)
(c) \( f(t) = C e^{-2t} \)
(d) \( f(t) = C \cos 2t \)
(e) None of the above

Answer: (c). Our particular solution should generally have the same form as the driving term.

6. The general solution to \( f'' + 7f' + 12f = 0 \) is \( f(t) = C_1 e^{-3t} + C_2 e^{-4t} \). What is a particular solution to \( f'' + 7f' + 12f = 5e^{-6t} \)?

(a) \( f(t) = \frac{5}{6} e^{-6t} \)
(b) \( f(t) = \frac{5}{31} e^{-6t} \)
(c) \( f(t) = \frac{5}{20} e^{-6t} \)
(d) \( f(t) = e^{-3t} \)
(e) None of the above

Answer: (a). We conjecture that a particular solution should be of the form \( f(t) = Ce^{-6t} \). We take derivatives finding that \( f'(t) = -6Ce^{-6t} \) and \( f''(t) = 36Ce^{-6t} \). We put these into the differential equation and find that \( 36C - 42C + 12C = 5 \) and thus \( C = \frac{5}{6} \).
7. To find a particular solution to the differential equation

\[ \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t, \]

we replace it with a new differential equation that has been “complexified.” What is the new differential equation to which we will find a particular solution?

(a) \[ \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{2it} \]
(b) \[ \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{3it} \]
(c) \[ \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it} \]
(d) \[ \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-2it} \]
(e) \[ \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-it} \]
(f) None of the above.

Answer: (c).

by Christopher Storm

DEQ.10.16.070 cks24
AU CS MA244 S08: 0/6/88/0/6/0

8. To solve the differential equation \[ \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it}, \] we make a guess of \( y_p(t) = Ce^{it}. \) What equation results when we evaluate this in the differential equation?

(a) \[ -Ce^{it} + 3Cie^{it} + 2Ce^{it} = Ce^{it} \]
(b) \[ -Ce^{it} + 3Cie^{it} + 2Ce^{it} = e^{it} \]
(c) \[ Cie^{it} + 3Ce^{it} + 2Ce^{it} = e^{it} \]
(d) \[ -Ce^{it} + 3Ce^{it} + 2Ce^{it} = Ce^{it} \]
9. To solve the differential equation \( \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it} \), we make a guess of \( y_p(t) = Ce^{it} \). What value of \( C \) makes this particular solution work?

(a) \( C = \frac{1 + 3i}{10} \)
(b) \( C = \frac{1 - 3i}{10} \)
(c) \( C = \frac{1 + 3i}{\sqrt{10}} \)
(d) \( C = \frac{1 - 3i}{\sqrt{10}} \)

**Answer:** (b).

10. In order to find a particular solution to \( \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t \), do we want the real part or the imaginary part of the particular solution \( y_p(t) = Ce^{it} \) that solved the complexified equation \( \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it} \)?

(a) Real part
(b) Imaginary part
(c) Neither, we need the whole solution to the complexified equation.

**Answer:** (a).
11. What is a particular solution to \( \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = \cos t \)?

(a) \( y_p(t) = \frac{3}{10} \cos t + \frac{1}{10} \sin t \)

(b) \( y_p(t) = -\frac{3}{10} \cos t + \frac{1}{10} \sin t \)

(c) \( y_p(t) = \frac{1}{10} \cos t - \frac{3}{10} \sin t \)

(d) \( y_p(t) = \frac{1}{10} \cos t + \frac{3}{10} \sin t \)

Answer: (d).

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DEQ.10.16.110 cks28
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AS DH MA3335 010 S12: 0/0/0/100 time 4:00
AS DH 3335 010 S13: 0/21/17/63 time 5:30
CC KC MA334 S13: 12/0/44/44 time 6:00
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