MathQuest: Differential Equations

Modeling with Systems

1. In the predator - prey population model

$$\begin{aligned}\frac{dx}{dt} &= ax - \frac{ax^2}{N} - bxy \\ \frac{dy}{dt} &= cy + kxy\end{aligned}$$

with $a > 0$, $b > 0$, $c > 0$, $N > 0$, and $k > 0$,

which variable represents the predator population?

(a) x

(b) $\frac{dx}{dt}$

(c) y

(d) $\frac{dy}{dt}$

Answer: (c). Since we have y' increasing due to interactions between x and y, y must be the predator in this system. We want to make sure that students know how to interpret the terms of a model. It would be good to discuss the form of the $\frac{dx}{dt}$ equation so students see that $ax - ax^2/N$ represents logistic growth.

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DEQ.10.22.010
CC MP MA334 S07: 0/12/**88**/0
CC KC MA334 S08: 0/6/**94**/0 time 3:00
AU CS MA244 S08: 5/5/**66**/24
CC KC MA334 S09: 12/0/**88**/0 time 3:00
CC KC MA334 S10: 0/0/**95**/5 time 2:30
CC KC MA334 S12: 0/5/**95**/0
AS DH MA3335 010 S12: 0/0/**100**/0 time 2:30
AS DH 3335 010 S13: 7/0/**67**/26 time 2:30
CC KC MA334 S13: 0/0/**100**/0 time 2:00
AS DH 3335 010 S14: 3/0/**75**/22 time 2:40
AS DH 3335 010 S15: 8/4/**88**/0 time 3:00

2. In which of the following predator - prey population models does the prey have the highest intrinsic reproduction rate?

(a)

$$\begin{aligned} P' &= 2P - 3Q * P \\ Q' &= -Q + 1/2Q * P \end{aligned}$$

(b)

$$\begin{aligned} P' &= P(1 - 4Q) \\ Q' &= Q(-2 + 3P) \end{aligned}$$

(c)

$$\begin{aligned} P' &= P(3 - 2Q) \\ Q' &= Q(-1 + P) \end{aligned}$$

(d)

$$\begin{aligned} P' &= 4P(1/2 - Q) \\ Q' &= Q(-1.5 + 2P) \end{aligned}$$

Answer: (c). The first task is to figure out which is the prey. In all models, we have a negative interaction term in the P' equation, so P must be the prey. The largest intrinsic growth rate will be the largest coefficient on the P term - which is 3 in system (c).

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DEQ.10.22.030
CC MP MA334 S07: 4/0/**88**/8
CC KC MA334 S08: 18/12/**71**/0 time 3:00
AU CS MA244 S08: 0/5/**90**/5
CC KC MA334 S09: 0/0/**100**/0 time 3:00
CC KC MA334 S10: 0/0/**95**/5 time 3:00
HC AS MA304 S11: 0/5/**85**/10/0/0
AS DH MA3335 010 S12: 0/0/**100**/0 time 1:50
AS DH 3335 010 S13: 0/30/**67**/4 time 3:40
CC KC MA334 S13: 0/0/**100**/0 time 2:45
AS DH 3335 010 S14: 9/6/**72**/13 time 3:00

3. For which of the following predator - prey population models is the predator most successful at catching prey?

(a)

$$\begin{aligned}\frac{dx}{dt} &= 2x - 3x * y \\ \frac{dy}{dt} &= -y + 1/2x * y\end{aligned}$$

(b)

$$\begin{aligned}\frac{dx}{dt} &= x(1 - 4y) \\ \frac{dy}{dt} &= y(-2 + 3x)\end{aligned}$$

(c)

$$\begin{aligned}\frac{dx}{dt} &= x(3 - 2y) \\ \frac{dy}{dt} &= y(-1 + x)\end{aligned}$$

(d)

$$\begin{aligned}\frac{dx}{dt} &= 4x(1/2 - y) \\ \frac{dy}{dt} &= 2y(-1/2 + x)\end{aligned}$$

Answer: (b). We must first decide which is the predator. We have a positive interaction term for the y' equation in each system, so y is the predator. We assume that most successful at catching prey would imply the greatest increase to y' occuring in the interaction, or the greatest decrease to x'. System (b) has the largest coefficients on both the y' and the x' interaction terms, 3 and -4 respectively.

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DEQ.10.22.040
AU CS MA244 S08: 5/**95**/0/0
CC KC MA334 S12: 0/**100**/0/0
HC AS MA304 S11: 0/**75**/5/20/0/0
CC KC MA334 S13: 0/**100**/0/0

4. In this predator - prey population model

$$\begin{aligned}\frac{dx}{dt} &= -ax + bxy\\ \frac{dy}{dt} &= cy - dxy\end{aligned}$$

with $a > 0$, $b > 0$, $c > 0$, and $d > 0$,

does the prey have limits to its population other than that imposed by the predator?

(a) Yes

(b) No

(c) Can not tell

Answer: (b). The prey in this case is y, since we have a negative interaction term. In the absence of interactions, the prey population grows without bound ($y' = cy$), thus, no there are no growth limits other than those imposed by the predator.

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DEQ.10.22.050
CC MP MA334 S07: 0/**83**/17
AU CS MA244 S08: 40/**55**/5
CC KC MA334 S09: 0/**88**/12 time 3:00 CC KC MA334 S10: 15/**70**/15 time 3:00
CC KC MA334 S12: 4/**92**/4
AS DH MA3335 010 S12: 0/**100**/0 time 2:00

5. In this predator - prey population model

$$\begin{aligned}\frac{dx}{dt} &= ax - \frac{ax^2}{N} - bxy\\ \frac{dy}{dt} &= cy + kxy\end{aligned}$$

with $a > 0$, $b > 0$, $c > 0$, and $k > 0$,

if the prey becomes extinct, will the predator survive?

(a) Yes

(b) No

(c) Can not tell

Answer: (a). Since we have a positive interaction term in the y' equation, y is the predator here. If we had a 0 population for the prey, we are left with $y' = cy$, and the predator's population will grow without bound.

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DEQ.10.22.060
CC MP MA334 S07: **92**/4/4

6. In this predator - prey population model

$$\begin{aligned} \frac{dx}{dt} &= ax - \frac{ax^2}{N} - bxy \\ \frac{dy}{dt} &= cy + kxy \end{aligned}$$

with $a > 0$, $b > 0$, $c > 0$, $N > 0$, and $k > 0$,

are there any limits on the prey's population other than the predator?

(a) Yes
(b) No
(c) Can not tell

Answer: (a). In this model, the prey must be x, since x' has a negative interaction term. If we factor x', we end up with $x' = ax(1-\frac{x}{N})-bxy$. In the absence of predators, we have logistic growth for the prey.

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DEQ.10.22.070
AU CS MA244 S08: **72**/28/0
HC AS MA304 S08: **85**/11/4
CC KC MA334 S09: **92**/8/0
AS DH MA3335 010 S12: **100**/0/0 time 1:20

7. On Komodo Island we have three species: Komodo dragons (K), deer (D), and a variety of plant (P). The dragons eat the deer and the deer eat the plant. Which of the following systems of differential equations could represent this scenario?

(a)

$$\begin{aligned} K' &= aK - bKD \\ D' &= cD + dKD - eDP \\ P' &= -fP + gDP \end{aligned}$$

(b)

$$\begin{aligned} K' &= -aK + bKD \\ D' &= -cD - dKD + eDP \\ P' &= fP - gDP \end{aligned}$$

(c)

$$\begin{aligned} K' &= aK - bKD + KP \\ D' &= cD + dKD - eDP \\ P' &= -fP + gDP - hKP \end{aligned}$$

(d)

$$\begin{aligned} K' &= -aK + bKD - KP \\ D' &= -cD - dKD + eDP \\ P' &= fP - gDP + hKP \end{aligned}$$

Answer: (b). Standard systems modeling question. This will take students a while.

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DEQ.10.22.080
CC MP MA334 S07: 4/**91**/0/4
AU CS MA244 S08: 0/**95**/0/5

8. In the two species population model

$$\begin{aligned} R' &= 2R - bFR \\ F' &= -F + 2FR \end{aligned}$$

for what value of the parameter b will the system have a stable equilibrium?

(a) $b < 0$
(b) $b = 0$
(c) $b > 0$
(d) For no value of b

Answer: (c). Understanding how to find and interpret equilibria

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DEQ.10.22.090
CC KC MA334 S09: 0/0/**85**/15 time 2:50

9. Two forces are fighting one another. x and y are the number of soldiers in each force. Let a and b be the offensive fighting capacities of x and y, respectively. Assume that forces are lost only to combat, and no reinforcements are brought in. What system represents this scenario?

   (a)
   $$\begin{aligned}\frac{dx}{dt} &= -ay \\ \frac{dy}{dt} &= -bx\end{aligned}$$

   (b)
   $$\begin{aligned}\frac{dx}{dt} &= -by \\ \frac{dy}{dt} &= -ax\end{aligned}$$

   (c)
   $$\begin{aligned}\frac{dx}{dt} &= y - a \\ \frac{dy}{dt} &= x - b\end{aligned}$$

   (d)
   $$\begin{aligned}\frac{dx}{dt} &= y - b \\ \frac{dy}{dt} &= x - a\end{aligned}$$

   Answer: (b). x will lose forces at a rate proportional to the size of y, with constant of proportionality equal to y's fighting efficiency, b. We find the equation for y similarly. This is the first in a sequence of five questions on the Lanchester Combat model.

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   DEQ.10.22.100
   CC KC MA334 S09: 19/**77**/4/0 time 3:00
   CC KC MA334 S10: 20/**75**/5/0 time 2:00
   CC KC MA334 S12: 30/**53**/4/13
   CC KC MA334 S13: 38/**58**/4/0

10. Two forces, x and y, are fighting one another. Let a and b be the fighting efficiences of x and y, respectively. Assume that forces are lost only to combat, and no reinforcements are brought in. How does the size of the y army change with respect to the size of the x army?

(a) $\frac{dy}{dx} = \frac{ax}{by}$

(b) $\frac{dy}{dx} = \frac{x}{y}$

(c) $\frac{dy}{dx} = \frac{y}{x}$

(d) $\frac{dy}{dx} = -by - ax$

Answer: (a). $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{ax}{by}$. It may be helpful to review derivatives of parametric equations before doing this problem. This is the second in a sequence of five questions on the Lanchester Combat model.

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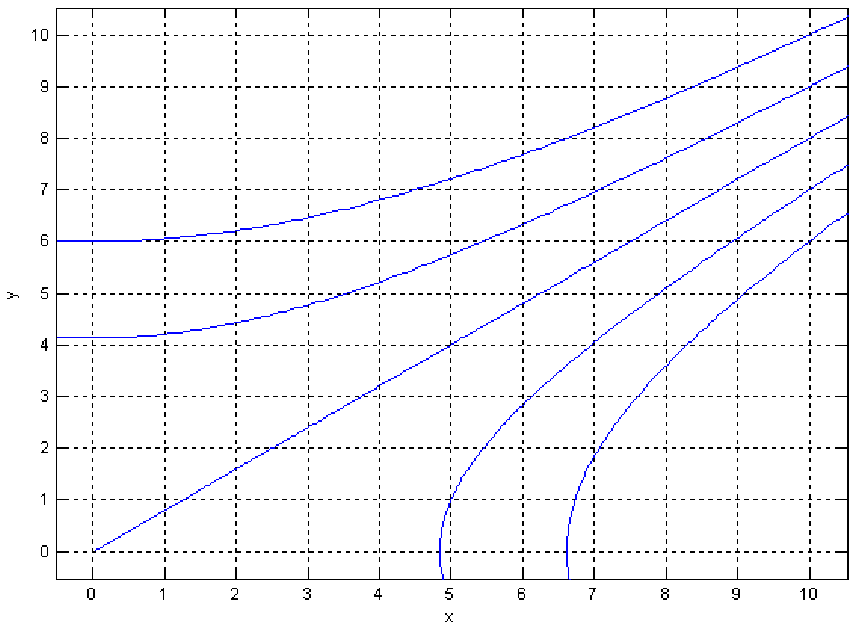
DEQ.10.22.110

11. Two forces, x and y, are fighting one another. Assume that forces are lost only to combat, and no reinforcements are brought in. Based on the phase plane below, if $x(0) = 10$ and $y(0) = 7$, who wins?



(a) x wins

(b) y wins

(c) They tie.

(d) Neither wins - both armies grow, and the battles escalate forever.

Answer: (a). With this starting condition, we find that x wins with about 5 units of strength remaining when y has been completely killed off. This question could be followed up by asking about other starting positions. Following the other trajectories shown, if we have $x(0) = 10$ and $y(0) = 8$ the battle ends in a tie. If $y(0) > 8$ for $x(0) = 10$ then y wins. This is the third in a sequence of five questions on the Lanchester Combat model.

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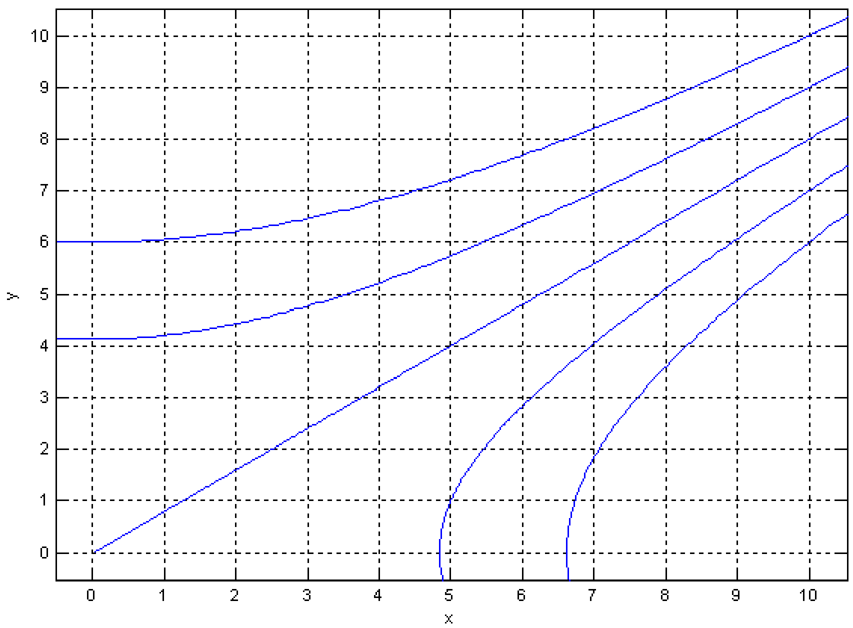
DEQ.10.22.120
CC KC MA334 S09: **42**/23/23/12 time 2:00
CC KC MA334 S10: **90**/5/5/0 time 2:30
CC KC MA334 S12: **88**/4/4/4
CC KC MA334 S13: **62**/0/23/15 time 2:20

12. Two forces, x and y, are fighting one another. Assume that forces are lost only to combat, and no reinforcements are brought in. Based on the phase plane below, which force has a greater offensive fighting efficiency?



(a) x has the greater fighting efficiency.

(b) y has the greater fighting efficiency.

(c) They have the same fighting efficiencies.

Answer: (b). If both forces started with equal numbers, the graph shows us that y would win. Thus, y must have the greater fighting efficiency. The explanation could also be tied directly to the slope of the "tie" line. This is the fourth in a sequence of five questions on the Lanchester Combat model.

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DEQ.10.22.130
CC KC MA334 S10: 0/**95**/5 time 1:30
CC KC MA334 S12: 22/**78**/0
CC KC MA334 S13: 15/**82**/4 time 2:30

13. Two forces, x and y, are fighting one another. Assume that forces are lost only to combat, and no reinforcements are brought in. You are the x-force, and you want to improve your chance of winning. Assuming that it would be possible, would you rather double your fighting efficiency or double your number of soldiers?

(a) Double the fighting efficiency

(b) Double the number of soldiers

(c) These would both have the same effect

Answer: (b). The "tie" line has the equation $y = \sqrt{a/b}x$, so we fare better by doubling $x(0)$ than by doubling a. (a and b are the fighting efficiencies of x and y, respectively.) This is the last in a sequence of five questions on the Lanchester Combat model.

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DEQ.10.22.140
CC KC MA334 S10: 15/**25**/60 time 2:00
CC KC MA334 S12: 48/**0**/52
CC KC MA334 S13: 19/**4**/77 time 2:30