Creating Discussions with Classroom Voting in Linear Algebra

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We present a study of classroom voting in linear algebra, in which the instructors posed multiple choice questions to the class and then allowed a few minutes for consideration and small group discussion. After each student in the class voted on the correct answer using a classroom response system, a set of clickers, the instructor then guided a class-wide discussion of the results. We recorded the percentage of students voting for each option on each question used in 18 sections of linear algebra, taught by 10 instructors, at 8 institutions, over the course of 5 years, together recording the results of 781 votes on a collection of 311 questions. To find the questions most likely to provoke significant discussions, we identify the six questions for which votes were most broadly distributed. Here we present these questions, we discuss how we used them to advance student learning, and we discuss the common features of these questions, to identify why they were so good at stimulating discussions.

Keywords: classroom voting; peer instruction; conceptests; linear algebra; clickers; classroom response systems

1. Introduction

Numerous studies have established the value of small group learning in mathematics and related disciplines (see e.g. [1]). Springer, Stanne, and Donovan [2] performed a meta-analysis in which they found that at the undergraduate level small group learning is "effective in promoting greater academic achievement, more favorable attitudes toward learning, and increased persistence through SMET (science, mathematics, engineering, and technology) courses and programs." However, designing activities to integrate small group learning into the undergraduate mathematics classroom on a regular basis is a significant challenge. Classroom voting is a teaching method that allows several brief episodes of small group work to be seamlessly integrated into an otherwise traditional class period.

In this teaching method, the instructor (1) poses a multiple-choice question tothe class, (2) allows a few minutes for consideration and small group discussion, (3) calls on each member of the class to vote on the correct answer, often with an electronic clicker, then (4) guides a class wide discussion of the results. This teaching method has several positive effects: (1) every student plays an active role in the lesson, (2) the instructor receives immediate feedback on the level of the students' understanding, and (3) students regularly participate in discussions about mathematics. It is this final point that is the emphasis of the study presented here.

Many studies [3-10] report the positive effects of using classroom voting in mathematics; common themes among these studies are that students enjoy the method and that it creates a positive and engaging learning environment. In particular, Zullo *et al.* [11] found through post-course surveys that overwhelming majorities of students reported that voting made class more fun and it helped them engage in the material, and further, if two sections of a class were offered, one with voting and one without, the majority of students would choose the section with voting. The Cornell GoodQuestions study [12] compared the exam scores from 17 parallel sections of differential calculus which were taught with a range of methods and found that sections using classroom voting showed significantly higher student exam scores, in comparison with sections taught in a more traditional manner, when voting was accompanied by pre-vote small-group discussions. This confirms the general results above about the power of small-group learning, and leads to our research question: What types of classroom voting questions are most likely to produce the most useful discussions?

2. Project MathQuest: Math Questions to Engage Students

Carroll College is a small private liberal arts college in Helena, Montana, which enrols about 1500 students. Thus, most of our mathematics courses are small, with perhaps 15 to 30 students. We began using classroom voting in fall 2003 in calculus, and we were consistently impressed at how powerful this method could be at engaging and involving students in each lesson. We found that if we used classroom voting regularly, with several votes interspersed throughout each class period, the students became experienced with the process and could quickly switch back and forth between lecture segments and voting segments.

We followed the four-step process for voting described in the introduction. During the post-vote discussion (step 4), we called on individual students by name and asked: “What did you vote for and why?” Although we did not use the computer to select students to be called on, we did our best to choose students at random, so that each student knew that he or she might be called on next, thus encouraging all to be attentive. After each student gave an explanation, we would ask the same question of another student, without giving any feedback as to whether the answer was right or wrong. In this way, we encouraged the students to figure out the right answer for themselves, deliberately trying not to reveal the right answer until after most of the students appeared to understand. During these post-vote discussions, we showed the students that it did not matter whether their vote was right or wrong, as long as they could explain it. The only unacceptable answer was “I just guessed.” If a student answered in that way, we would remind the class that this answer was not sufficient, and call on that particular student first after the next vote. We found that the knowledge that one might be called on in the class-wide discussion provided sufficient motivation for students to take the small-group discussion phase (step 2) seriously, with no need to grade them on their participation.

Once the students invested themselves in the process, they reported that classroom voting was a lot more fun than regular classes, saying that it made class "go faster," and they even complained when we had an occasional period with no voting. Student comments suggest that they enjoyed not only the clickers and voting itself but the discussions that accompanied it as well.

As a result of this very positive response from students, and our own view of this as a valuable teaching method, we decided to develop the questions necessary to using voting in our linear algebra and differential equations classes, which we pursued with the support of a grant from the National Science Foundation as "Project MathQuest." We developed a collection currently containing 311 multiple-choice classroom voting questions for linear algebra and 350 questions for differential equations, both of which are freely available on our website (http://mathquest.carroll.edu).

In order to study the effects of these questions, we recorded the percentage of the class voting for each option whenever a question was used. When we compared voting results on a particular question when posed to two different classes, we found that usually the results were similar, and thus that having records of previous votes could give a useful indication of what types of student responses to expect. Sometimes we found that two classes would vote very differently on the same question, most often when in one class a question was asked early in the lesson, to help students explore a new idea, while in the other class the question was asked later in the lesson, to assess student understanding.

As we tested these questions, we found that many produced little if any discussion, either small-group pre-vote discussions or post-vote class-wide discussions. Other questions regularly stimulated excellent discussions, with students bringing up a variety of different perspectives, helping students to analyze the mathematical concepts in substantial depth. We observed that questions in which a strong majority of the class voted correctly did not usually produce much discussion. These questions often served a useful purpose, to quickly confirm student understanding, and to give them a first practice with a new idea. However, this type of voting pattern indicated that a question would be unlikely to produce significant discussion, and thus little small-group collaboration. Instead, we noticed that questions which provoked a significant percentage of students to vote for several different options were more likely to be a fertile background for discussion. This certainly makes sense: If everyone agrees that the answer is clear, then what is there to talk about? However, if there are several answers which are appealing, each attracting a significant number of student votes, then it is much more likely that during a pre-vote discussion, a particular small group will contain students who will express contrasting opinions, and thus have something to talk about. As a result, we began the current study, gathering voting results, and analyzing them in order to identify the questions most likely to produce diverse votes and thus significant discussions. Occasionally we would find a question that was poorly worded and, although it might produce discussion, it was clearly not helpful in the classroom. In these cases, we would revise the question in order better achieve our goals.

As a caveat to the analysis that follows, it is important to note while we want to find questions where students will bring differing opinions to the small-group discussions, the votes and thus the data that we recorded took place after the small-group discussions. In class, we observed that the small groups tended to form a consensus and then vote uniformly. Thus, when we record votes that are broadly distributed among the options presented, these are questions for which the small groups reached different conclusions after their discussions. To produce good small group discussions, we want students to form differing conclusions before the discussions. We base this study on the hypothesis that there is a strong overlap between these two types of questions: We expect that questions which regularly cause different small groups to reach different conclusions are usually questions which also cause individuals to reach different conclusions before any discussion, and thus that these questions will be likely to produce rich and useful small group discussion. Our qualitative observations tend to confirm this hypothesis. In the future, it may be useful to test this hypothesis by conducting two votes on each question, one preceding and one following the small group discussions. However, for the current study, we decided that in order to make the most efficient use of class time, we would only conduct one vote on each question.

3. Data Collection and Analysis

We gathered a collaboration of colleagues at institutions across the US who all agreed to use our voting questions and report the voting results. Over the past five years, we received voting results from 18 sections of linear algebra, taught by 10 instructors, at 8 institutions, recording a total of 781 votes. Each of these results has been incorporated into the teacher's edition of our question collection, which is freely available with an e-mail to the authors. These institutions include three small private liberal arts colleges (Carroll, Hood, and Kenyon), one small private university (Walla Walla), two community colleges (Middlesex County College and Spokane Falls), and two high schools (Helena and Capitol) which were both offering linear algebra for college credit.

After compiling this collection of voting results, we decided to only consider questions for which we had results from at least five classes, which limited us to 66 questions. In order to find potentially good discussion questions, based on these past votes, we decided to look for questions where the votes were spread most widely. Ideally a question would not produce a majority voting for any particular option. For each vote, we identified the option which is the winner, the option receiving more votes than any other, and recorded the percentage voting for the winner. Then for each question, we computed the average winning percent among all of the classes who voted on this question. For example, suppose that we have the voting results for a particular question from two classes: In one class 40% voted for (a), 35% for (b), and 25% for (c), and in the other class 25% voted for (a), 25% for (b), and 50% for (c). The winner in the first class is (a) with 40%, the winner in the second class is (c) with 50%, and so the average winning percent is 45%.

We then ranked each of the 66 questions based on their average winning percent. The question with the lowest value had an average winning percent of 48%, the question with the highest value had an average winning percent of 99%, and the median average winning percent was 74%. The questions with the smallest average winning percent were those where the votes were most widely spread, and when we reviewed them, we found that in our recollections, most of these questions did indeed produce very good discussions. Here we present the six questions which had the smallest average winning percent.

4. The Six Questions with Most Widely Distributed Votes

Figure 1 shows the question with the lowest average winning percent, indicating the most widely distributed votes. This question is from our section on linear independence and is designed to be asked after students have learned how to test whether a set of vectors is linearly independent by forming a matrix with these vectors as the columns and putting this matrix into reduced row echelon form. Students have learned that if each column contains a leading one, then the original set of vectors was linearly independent. The scenario presented here is that we were given four vectors (v_1, v_2, v_3 and v_4), we created a matrix using these vectors as the columns, and we put this matrix into reduced row echelon form. The students are not given the vectors or this initial matrix, but instead only receive the matrix in reduced row echelon form, where we see that only columns 1 and 2 contain leading ones, while columns 3 and 4 contain other numbers. The students are then asked how we can write the fourth vector as a linear combination of the others.

This question can be approached in several different ways. We can interpret the matrix as an augmented matrix to solve the system $c_1 v_1 + c_2 v_2 + c_3 v_3 = v_4$, where c_1, c_2 and c_3 are scalar coefficients. Based on this view, the two rows of the matrix are interpreted as $c_1 + 2\,c_3 = 1$ and $c_2 + 3\,c_3 = 1$. We recognize that c_3 can be a free parameter, and so if we set $c_3 = 0$, then $c_1 = c_2 = 1$, and thus $v_4 = v_1 + v_2$, which is option (a). Alternatively, we can interpret the matrix as the coefficient matrix for the homogeneous system $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$. In this case, the rows are interpreted as $c_1 + 2\,c_3 + c_4 = 0$ and $c_2 + 3\,c_3 + c_4 = 0$, and so we have two free parameters. Our goal is to solve for v_4 and so we set $c_4 = 1$, which then brings us to the system above. Students may vote for (b) if they interpret the rows of the matrix not as expressing relations between the vector coefficients, but between the vectors themselves, thus erroneously interpreting the first row as $v_1 + 2\,v_3 + v_4 = 0$.

Table 1 shows the results of five votes on this question, clearly demonstrating that significant numbers of students are attracted by all four options. Only an average of

46% of students vote for the correct answer (a), and option (c) occasionally wins the vote as well. The average winning percent was only 48%, the lowest value among all of the 66 questions that we analysed.

The post-vote discussions for this question were particularly rich, and students brought up a wide variety of misconceptions. In guiding these discussions, after students explained how they interpreted the rows, we wrote these up on the board, both correct interpretations ($c_1 + 2\ c_3 = 1$ and $c_1 + 2\ c_3 + c_4 = 0$) and well as incorrect ones ($v_1 + 2\ v_3 + v_4 = 0$). When explicitly written down, students could usually identify the erroneous equations fairly quickly, but they had more difficulty recognizing that the two other interpretations were both right. These discussions did consume a substantial quantity of class time, roughly 4 minutes for the small group discussions before the vote, and even longer for the class-wide discussion after the vote. However, the depth of the discussion indicated to us that the students were learning a great deal from this question, and so the time was well spent.

Question 2 (Figure 2) poses a very broad question, rather than a specific numerical case. Because the determinant of matrix *A* is zero, this means it is not invertible, and thus that there is not one unique solution to $Ax = b$. There are some cases for which there would be no solution, and other cases for which there would be infinitely many solutions, and thus the best answer is (d). Table 2 shows that (b) is the least popular answer, indicating that most students recognize that this system will not have one unique solution. However whether we have no solution or infinitely many solutions is not clear. In leading the post-vote discussions of a very general question like this, after a student explains their vote, it can be useful to explicitly ask for a supporting example: "Can you give an example of a matrix *A* and a vector *b* so that we have (no solution, one solution, infinitely many solutions)?"

Question 3 (Figure 3) is designed to be asked early in the lesson on vector spaces, asking students to apply the definition of a vector space to the set of 2 x 2 matrices with determinant equal to zero. Answer (d) correctly provides a counterexample for why this set is not a vector space, providing two matrices with zero determinant, which when summed produce a matrix which has a non-zero determinant, thus demonstrating that this set is not closed under matrix addition. Even this answer, however, is not fully complete, as it does not begin by showing that the starting matrices have determinant equal to zero. Thus this question also provides an opportunity to discuss writing careful answers to mathematical questions. As we see in Table 3, in only three of the seven sections did a majority of students vote correctly, and options (a), (b), (c), and (d) were all winners in different sections.

Question 4 (Figure 4) asks students about the invertibility of a coefficient matrix, given that the homogenous equation $A\boldsymbol{x}=\boldsymbol{0}$ has only the trivial solution. This means that "(b) Matrix *A* has an inverse." As we see in Table 4, this is the first question that we have considered where one of the distracters (c) is just as popular as the correct answer (b). Further, even though the question has only three options, in only four of the eight sections did any of the options receive a majority of votes. Students were remarkably evenly split between voting that (a) *A* has no inverse, that (b) *A* has an inverse, and that (c) this tells us nothing about the invertibility of *A*. We usually ask this question immediately after introducing matrix inverses, and we use this to begin to make the connection between existence of the inverse and the number of solutions to a corresponding system of equations. The class-wide discussion focuses on helping students to make the logical connections necessary for understanding the relationship between these ideas.

Figure 5 shows the question with the next lowest average winning percent. This is the very first question in our section on "Linear Transformations and Projections" and it was deliberately written so that it could be asked with no introduction to transformations. In general, we have found that some of the best discussions result when we provide as little introduction as possible before posing a question, and instead let the students discover the key ideas as they explore the question. In this case, students can test a few vectors, finding that this transformation leaves (1 0) unchanged, but turns (0 1) into (0 -1), and thus this is a transformation which (b) reflects vectors across the x_1 axis. Some students may also notice that the images of these two important vectors are in fact the two columns of the transformation matrix, and they may wonder if this is a coincidence.

As we see in Table 5, the correct answer is usually the winner, but substantial numbers of students regularly vote for (a) and (c). In the class-wide discussion that follows this question, we often find that many students are trying to reason directly from the matrix itself, which is not always easy to do. Others students are trying out this transformation on test vectors, but the test vectors which they select are not always the most useful ones: In one case a student found that (1, 1) is turned into (1, -1), and thus could not tell whether the vector had been reflected about the x_1 axis, or whether it had been rotated about the origin by $\pi/2$. The graphical nature of this question makes it especially useful, because it helps students make the connection between different representations of mathematical objects. This is a question where it can be useful to explicitly instruct students to use diagrams in their small group discussions.

In Question 6 (Figure 6), students must identify a graph of a line representing the null space of a given 2 x 2 matrix. In order to work through this problem, students must first solve for the null space, multiplying this matrix by a general vector (x y) to get the equation $x + 2y = 0$, or $y = -\frac{1}{2}x$. Alternately, students can solve the system $A\boldsymbol{x}=\boldsymbol{0}$, to find that the null space consists of all scalar multiples of the vector (-2, 1). Four lines are presented to the students on a graph, and all go through the origin, so they all represent valid subspaces. The lines have slopes of 2, $\frac{1}{2}, -\frac{1}{2}$ and – 2 (for answers (a), (b), (c), and (d) respectively) and thus they probe for possible sign errors and inversion errors in the solution. Further, the scales of the *x* and *y* axes of the graph are not the same ($\Delta x = 1$, $\Delta y = 2$), so the graph requires some consideration. As we see in Table 6, the correct answer (c) is usually the winner, however (d) receives a significant number of votes as well, indicating that most students successfully find that the slope must be negative, but that common errors occur in recognizing the magnitude of this slope, either in the algebra or in reading the graph. In leading post-vote discussions of this question, we found it useful to explicitly write out student work on the board, as they verbally described the process of solving for the null space and finding the slope of this line. Once the slope of the line is determined, then students quickly recognize which line is the correct one.

5. Discussion and Conclusions

In this study we have used voting results in order to identify questions that are likely to generate useful discussions in linear algebra. Further, the ideas and methods used here could be applied to classroom voting questions and results on any topic.

We recall having very good post-vote discussions based on all six of the questions presented here, under most conditions. While no questions will produce excellent discussions for all classes in all situations, in our judgment, these six questions are well worth the investment of class time, and thus we encourage you to try them, whether or not you use voting on a regular basis.

It is important to note that, while goal of this study is to identify questions which will stimulate students to bring contrasting opinions to pre-vote small-group discussions, the data that we have analysed was collected after these discussions had taken place. Thus this study rests on the hypothesis that questions which lead student votes to be widely distributed after small-group discussions are usually questions which also cause individuals to reach different conclusions before any discussion, and thus that these questions will be likely to produce rich and useful small group discussions.

So, what are the characteristics of these six discussion questions? First, these are all questions which students find to be difficult. Easy questions rarely produce good discussion, because if all the students agree on the right answer, there is little to talk about. As a whole, these questions ask about very general issues. There are a great many questions in our collection that require students to perform specific calculations to get an answer. Among these six questions, while specific calculations may be required, finding the correct answer requires us to interpret the meaning of these calculations more broadly. These questions thus bring students away from the mechanics of calculation, and pose them with challenges relating to conjectures and theorems. This is one of the most difficult, and yet most essential features in the study of linear algebra. Thus, these questions demonstrate that the multiple-choice questions used in classroom voting can be used for far more than just superficial evaluation of computation, and instead can help students begin to grapple with the deeper theoretical issues. Further, as students discuss these subjects, dealing with other perspectives, these questions can help them learn to listen to different chains of mathematical reasoning, preparing them to create proofs of their own.

The remaining 60 questions for which we have votes from at least five classes are very diverse. Many of them require students to perform straight-forward computations, practicing a new method while in class. Others questions ask students to translate ideas between different representations, e.g. identifying the graph that corresponds to a given linear equation. A few of the questions were designed to provoke common errors or misconceptions, thus drawing a majority of students to vote for a particular incorrect answer. While questions of that type did not usually produce significant small-group pre-vote discussions, they were quite valuable teaching tools, as the students were surprised and intrigued to discover their error, thus creating a memorable learning event.

Some of the remaining 60 questions focused on broad, theoretical issues, much like the six questions identified and discussed above. Not all questions designed to probe theoretical issues were successful in either producing widely spread votes or significant discussions. We often found it difficult to anticipate which questions would engage the class in active discussion, and which produce a more muted response. However, we did find that questions which produced good discussions in the past were likely to be effective in the present. Thus, we found that a record of past votes can be a very valuable tool when creating a lesson plan. Further, we have begun including annotations in the teacher's edition of our question collection, pointing out questions which were particularly useful, and the conditions under which they were used.

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Suppose you wish to determine whether a set of vectors $\{v_1, v_2, v_3, v_4\}$ is linearly independent. You form the matrix $A = [v_1, v_2, v_3, v_4]$, and you calculate its reduced row echelon form, $R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. You now decide to write v_4 as a linear combination of v_1, v_2 and v_3. Which is a correct linear combination?

(a) $v_4 = v_1 + v_2$

(b) $v_4 = -\,v_1 - 2\,v_3$

(c) v_4 cannot be written as a linear combination of v_1, v_2 and v_3.

(d) We cannot determine the linear combination from this information

Figure 1. Question 1, the question for which the votes were most widely distributed.

	a)	**b)**	**c)**	**d)**	**Winner**	**Section**
	36%	36%	0%	27%	36% (a/b)	9
	52%	15%	0%	33%	52% (a)	10
	24%	19%	38%	19%	38% (c)	12
	38%	24%	5%	33%	38% (a)	16
	78%	15%	7%	0%	78% (a)	17
Avg.	**46%**	22%	10%	22%	48%	
STD	**21%**	9%	16%	14%	18%	

Table 1. Voting results from Question 1, with the correct answer indicated by the bold column.

Suppose that the determinant of matrix A is zero. How many solutions does the system $Ax = b$ have?

(a) 0

(b) 1

(c) Infinite

(d) Not enough information is given.

Figure 2. Question 2.

	a)	**b)**	**c)**	**d)**	**Winner**	**Section**
	27%	0%	27%	**45%**	45% (d)	3
	31%	11%	31%	**23%**	31% (a/c)	4
	29%	0%	36%	**36%**	36% (c/d)	5
	0%	0%	18%	**82%**	82% (d)	6
	6%	9%	36%	**49%**	49% (d)	11
	17%	21%	38%	**25%**	38% (c)	17
	12%	0%	24%	**65%**	65% (d)	18
Avg.	17%	6%	30%	**46%**	49%	
STD	12%	8%	7%	**21%**	18%	

Table 2. Voting results from Question 2, with the correct answer indicated by the bold column.

The set of all 2 x 2 matrices with determinant equal to zero is not a vector space.

Why?

(a) 2 x 2 matrices are not vectors.

(b) With matrices, *AB* need not equal *BA*.

(c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ is not in the set.

(d) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not in the set.

(e) None of the above

Figure 3. Question 3.

	a)	**b)**	**c)**	**d)**	**e)**	**Winner**	**Section**
	0%	5%	0%	**55%**	40%	55% (d)	3
	76%	12%	0%	**12%**	0%	76% (a)	7
	13%	30%	17%	**17%**	22%	30% (b)	9
	0%	0%	27%	**72%**	0%	72% (d)	12
	21%	29%	29%	**21%**	0%	29% (b/c)	14
	6%	28%	11%	**56%**	0%	56% (d)	15
	16%	16%	24%	**32%**	12%	32% (d)	17
Avg.	19%	17%	15%	**38%**	11%	50%	
STD	26%	12%	12%	**23%**	16%	20%	

Table 3. Voting results from Question 3, with the correct answer indicated by the bold column

We find that for a square coefficient matrix *A*, the homogeneous matrix equation

$AX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, has only the trivial solution $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This means that

(a) Matrix *A* has no inverse.

(b) Matrix *A* has an inverse.

(c) This tells us nothing about whether *A* has an inverse.

Figure 4. Question 4. If a homogeneous system has only the trivial solution, this means that (b) matrix *A* has an inverse.

	a)	**b)**	**c)**	**Winner**	**Section**
	18%	**68%**	14%	68% (b)	1
	5%	**63%**	32%	63% (b)	3
	7%	**15%**	78%	78% (c)	4
	50%	**17%**	33%	50% (a)	5
	38%	**24%**	38%	38% (a/c)	8
	21%	**39%**	39%	39% (b/c)	11
	20%	**30%**	50%	50% (c)	16
	29%	**42%**	29%	42% (b)	17
	32%	**32%**	37%	37% (c)	18
Avg.	24%	**37%**	39%	52%	
STD	14%	**19%**	18%	15%	

Table 4. Voting results from Question 4, with the correct answer indicated by the bold column.

Define $T(v) = Av$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $T(v)$

(a) reflects v about the x_2-axis.

(b) reflects v about the x_1-axis.

(c) rotates v clockwise $\pi/2$ radians about the origin.

(d) rotates v counterclockwise $\pi/2$ radians about the origin.

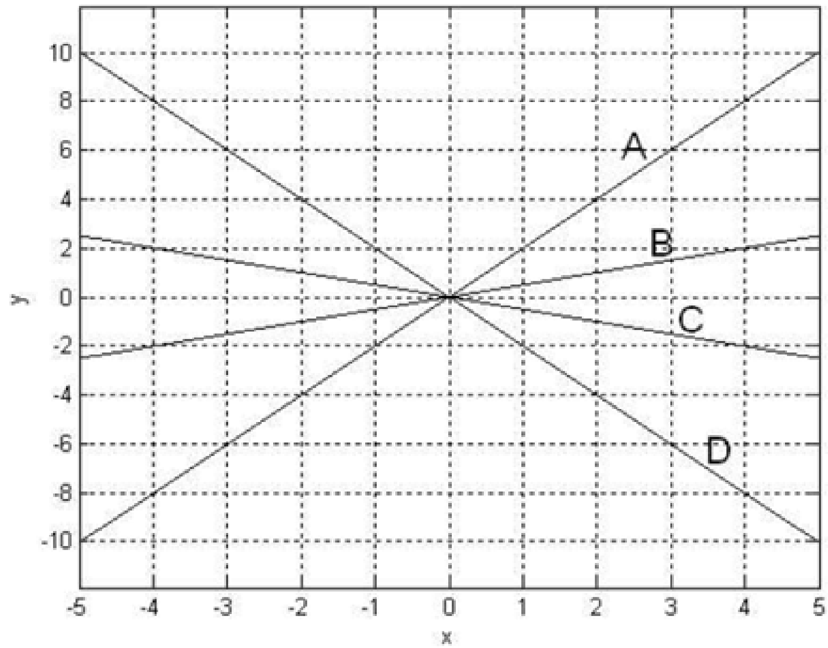
(e) None of the above

Figure 5. Question 5.

	a)	**b)**	**c)**	**d)**	**e)**	**Winner**	**Section**
	29%	**43%**	19%	5%	5%	43% (b)	3
	24%	**47%**	18%	12%	0%	47% (b)	7
	11%	**53%**	16%	21%	0%	53% (b)	9
	19%	**38%**	43%	0%	0%	43% (c)	12
	11%	**74%**	16%	0%	0%	74% (b)	16
	22%	**52%**	4%	11%	11%	52% (b)	17
Avg.	19%	**51%**	19%	8%	3%	52%	
STD	7%	**13%**	13%	8%	5%	12%	

Table 5. Voting results from Question 5, with the correct answer indicated by the bold column.

Which line in the graph below represents the null space of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$?



(a) line A

(b) line B

(c) line C

(d) line D

(e) None of the above

Figure 6. Question 6.

	a)	b)	c)	d)	e)	Winner	Section
	18%	12%	**23%**	18%	29%	29% (e)	7
	0%	0%	**59%**	26%	15%	59% (c)	9
	0%	0%	**75%**	17%	8%	75% (c)	12
	0%	0%	**67%**	33%	0%	67% (c)	13
	7%	14%	**46%**	21%	11%	46% (c)	17
Avg.	5%	5%	**54%**	23%	13%	55%	
STD	8%	7%	**20%**	7%	11%	18%	

Table 6. Voting results from Question 6, with the correct answer indicated by the bold column.