ADDRESSING COMMON STUDENT ERRORS WITH CLASSROOM VOTING IN MULTIVARIABLE CALCULUS

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BIOGRAPHICAL SKETCHES

Kelly Cline is an Associate Professor of Mathematics at Carroll College in Helena, Montana. His interests include innovative teaching methods and technologies, including classroom voting and online homework systems. He is PI on the NSF-Funded Project MathVote to study the use of classroom voting in collegiate mathematics.

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Holly Zullo is an Associate Professor of Mathematics at Carroll College, in Helena, Montana. She has been using classroom voting in calculus and other courses for several years, and she was PI on the NSF-funded Project MathQuest to develop voting questions for linear algebra and differential equations.

Ann Stewart is an Assistant Professor of Mathematics at Hood College, in Frederick, Maryland. She has used classroom voting in differential equations, linear algebra, and multivariable calculus, and she is one of the senior personnel on the NSF-funded Project MathVote.

Keywords: Multivariable calculus, classroom voting, peer instruction

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Abstract: One technique for identifying and addressing common student errors is the method of classroom voting, in which the instructor presents a multiple choice question to the class, and after a few minutes for consideration and small group discussion, each student votes on the correct answer, often using a hand-held electronic clicker. If a large number of students have voted for one particular incorrect answer, the instructor can recognize and address the issue. In order to identify multiple choice questions which are especially effective at provoking common errors, we recorded the percentages of students voting for each option on each question used in eleven sections of multivariable calculus, taught by four instructors, at two small liberal arts institutions, all drawing from the same collection of 317 classroom voting questions, over the course of five years, during which we recorded the results of 1,038 class votes. We found six questions in which, on average, more than 50% of each class voted for the same incorrect answer. Here we present these six questions and we discuss how we used them in the classroom in order to promote discussion and student learning.

Keywords: Multivariable calculus, classroom voting, peer instruction

1 INTRODUCTION

Classroom voting, sometimes called “Peer Instruction,” has developed a strong record of success as a teaching method in the university classroom throughout the STEM disciplines (see e.g. [2, 6, 8, 12, 13]). In this pedagogy, the instructor presents a multiple-choice question to the class, gives the students a few minutes for consideration and small group discussion, and then calls on them to vote on the correct answer, perhaps with a hand-held electronic clicker or perhaps through some more basic method, such as having students raise hands. After the vote, the instructor can guide a class-wide discussion of the question, asking students to explain their votes, and helping them to work out the correct answer. This teaching method requires all the students to participate in a small group discussion, to form an opinion, and to register this opinion. The votes themselves further serve as a formative assessment, giving the instructor an indication of student understanding, and they provide feedback to the students as well, when the correct answer is discovered through the post-vote discussion. There have been many positive reports on the effects of classroom voting in the collegiate mathematics classroom, indicating that this method can be successful in engaging the students with important mathematical ideas (see e.g. [1, 7]), that it can improve student exam scores [9], and that students report enjoying the process and prefer this interactive teaching method to traditional lectures [15].

One way in which classroom voting can be particularly effective is when a question provokes a majority of students to vote for the same incorrect answer. If this were to happen on a test, or other summative assessment, such a result would be concerning. However, as noted by Bruff [2], in a classroom voting context, this can create an excellent learning environment, which he calls a “time for telling.” The students anticipate that the majority will be correct, and thus they are usually surprised and intrigued when they find that the class as a whole is in error. Because the students have invested themselves in the learning process, they pay especially close attention to the resolution of the problem at hand.

However, it is often quite difficult to write a question which will successfully have this effect. Occasionally the instructor knows of a particular misconception or error that many students share, and knows exactly how to provoke this issue with a multiple choice question. However, more often, it is challenging to write a really good question which will get a majority of students to vote for a particular incorrect answer. Students are often distracted by issues which the instructor did not anticipate, so that the question does not have the focus that was intended. At the same time we have seen questions effectively provoke common errors in class, even though they were not written for this specific purpose. Often, the only way to identify good common-error questions is to test the questions out in a variety of classes, so that the voting results can indicate which questions succeeded in this goal, a process which we first applied on the topic of differential calculus [3], where we called these questions “misconception magnets.”

The purpose of the study presented here is to isolate a group of voting questions which are effective at provoking students to make common errors in multivariable calculus, and to discuss how these questions can be used in the classroom as a powerful platform for addressing those errors. To that end, we recorded the percentages of students voting for each option on each question used in eleven sections of this class taught over the course of five years.

2 CLASSROOM VOTING IN MULTIVARIABLE CALCULUS

We first used classroom voting in multivariable calculus during the fall of 2004 at Carroll College, in Helena, Montana. Carroll is a small liberal arts college of approximately 1,400 students, where multivariable calculus is required of all mathematics, engineering, and chemistry majors, which are the principal constituents of these classes. A typical section enrolls about 15 to 25 students. The voting questions that we used were drawn from several sources, including the collection of “ConcepTests” written to accompany the Hughes-Hallett *et al.* calculus text [5], which was the main text for the course, as well as a collection of 89 questions written by Mark Schlatter [11], and a number of questions that we wrote ourselves.

When we introduced voting, we were consistently impressed not only with the power of this teaching method to create an active learning environment, but also with how positively and enthusiastically the students reacted. We used voting several times in almost every 50 minute class period, which consumed a substantial quantity of class time, often about five minutes total for each question’s pre-vote consideration and discussion, the vote itself, and the post-vote class-wide discussion. However, we learned to use voting to get the students to work through examples that we would otherwise have done on the board. Although, using a less interactive teaching method, we would have been able to do a wider range of examples for the class, we decided that the level of student engagement in the examples that we did pursue more than made up for this reduction.

As we taught multivariable calculus with classroom voting, we also learned that it was helpful if the votes and discussion were not a summative assessment for the instructor's benefit, so that the votes did not constitute any part of the students' grades. Instead, we used the voting and discussion to teach the material, and so they formed an essential part of each lesson. There were many key ideas which we presented principally through the classroom voting process, thus students could see that the classroom voting pedagogy was for their benefit, to help them learn the material. Several studies have documented that the success of voting can hinge on whether students recognize that the voting is being conducted principally as a teaching tool, for their benefit, rather than as a time saving assessment technique for the instructor [4, 14].

Beginning in fall 2006, two of us (KC and MP) began recording the percentages of students voting for each option on the questions that we selected, in order to facilitate comparison between our sections, and to provide some evaluation of the effectiveness of the questions that we were using. We gathered this data for four years, then during the fifth year, we expanded the study, including a section taught at Carroll College by another instructor (HZ), as well as a section taught by a fourth instructor (AS) at Hood College, another liberal arts college. This permitted us to examine whether our voting results were dependent on the two instructors involved, or whether these questions would provoke similar results in other, similarly structured classes.

As we see in Table 1, the data analyzed here are from eleven sections of multivariable calculus, taught by four instructors, at two institutions, over the course of five years, all drawing from the same collection of classroom voting questions, including results from a total of 1,038 class votes. At the beginning of this study, our collection contained approximately 250 classroom voting questions, but we have been steadily adding to the collection, and it currently contains a total of 317 questions. A "student's edition” of this collection is freely available on the web (http://mathquest.carroll.edu), and a “teacher’s edition” with past voting results, solutions, and commentary is available with an e-mail request to the authors.

3 IDENTIFYING COMMON-ERROR QUESTIONS FROM VOTING RESULTS

To identify questions which consistently provoke a majority of students to vote for a particular incorrect option, we limited our analysis to questions for which we had results from at least five of the eleven sections, and which included data from at least two different instructors. We found that 81 of the 317 questions met these criteria.

We then computed the mean percentage of students voting for each option. We considered each section to be a single data point, so we did not weight this mean according the enrollment of each section. Instead we computed the arithmetic mean of the percentage of students voting for each option across each section reporting data for each question. In Figure 1, for each of these 81 questions, we plot the standard deviation of the percentage of students voting correctly versus the mean percentage of students voting correctly. This graph shows that a majority of the students voted correctly on most of the questions used. This is to be expected, as instructors will only pose a question if they expect that the students have the necessary preparation to answer it correctly. As a result, when the majority of students votes incorrectly, it is surprising and interesting to the students.

Next, we ranked the 81 questions, based on the average percentage of students voting for the most popular incorrect option, finding only six questions where, on average, a majority of students voted for the same incorrect answer. This tells us that it is quite rare to find a question which regularly provokes the majority of a class to vote for the same incorrect option, indicating that creating and identifying questions of this nature is a substantial challenge. These six questions are numbered in Figure 1, and presented in Figures 2 – 7, with complete voting results presented in Tables 2 – 7.

In these tables, note that the section numbers listed in the last columns of Tables 2 – 7 correspond to the section numbers listed in the first column of Table 1. Thus, for example, the first voting result presented in Table 2, stating that 21% voted for (a) and 79% voted for (b), comes from section 2. When we look at Table 1, we see that section 2 was taught in the fall of 2006 and enrolled 20 students.

4 THE SIX COMMON-ERROR QUESTIONS

In Question 1 (Figure 2) students know that the gradient vector is perpendicular to something, but they erroneously think that this perpendicularity applies to the surface plot of the function, and therefore answer that this statement is true. This question provides a useful opportunity to encourage students to classify the types of mathematical objects we are considering. When they realize that here the gradient is a two dimensional vector, and that the surface resides in three dimensions, it becomes clear that these will not necessarily be perpendicular. Table 2 presents the voting results, and although they include data from all four instructors, they show the most consistent voting patterns of the six questions presented, with smaller standard deviations than any of the others, indicating that this is a particularly robust error.

In Question 2 (Figure 3) the student error comes not from calculus, but from a failure to note how the variables are defined. In this business scenario, the total value of the company's equipment V is defined in thousands of dollars, so when the equipment increases in value by \$20,000, $dV = 20$. However, most students instead use $dV = 20{,}000$, leading to answer (a). It is certainly a valuable lesson to pay careful attention to how the variables are defined. As we see in Table 3, this question produced remarkably consistent results from three different instructors, probably because the common error being provoked here is not directly relevant to the topic being taught. Thus it is not particularly important whether the instructor poses this question early or late in the lesson, after many examples have been performed. In only one of the seven sections did a majority vote correctly. Students may initially see this as a "trick question," and so we have found that is it important for the instructor to provide positive guidance to the students, reminding them why it is important to think carefully about how variables are defined whenever we use mathematics in the real world.

Question 3 (Figure 4) asks students to change the order of integration of a double integral. To do this, they must first determine the geometrical region described by the limits of the given integral, and then recognize how to describe this region with the integrals posed in the opposite order. It is interesting to observe in Table 4, that in four of the sections, a strong majority votes erroneously for (d), while in the two other sections, more than 90% of both classes vote correctly, leaving only a small minority to vote for (d). As a result the standard deviation of the percentages of the classes voting for (d) is 44%, which is larger than any of the other six questions presented here.

Note particularly the difference between section 5 and section 6, which were both taught by the same instructor during the same semester in consecutive periods. In section 5, the question was only posed at the end of the lesson, after we had carefully discussed a process for changing the order of integration, and thus the question served to confirm the students' comprehension. In section 6, the instructor used the question at the beginning of the lesson, to introduce the subject. After the vote, the students were asked to sketch the regions of integration represented by the limits of the initial integral, and by answer (d). This produced a dramatic moment when the class realized that (d) was in error, which served to motivate a careful discussion how to change the order of integration. In section 10, this question was used after the students had already been working with integrals for a couple of days, so the class was no longer at an introductory level with this material.

Together, these results indicate that student understanding of this topic is fairly binary. Either students understand that changing the order of integration requires careful consideration, and probably a sketch of the region involved, or they erroneously think that this can be done on a more intuitional level.

Question 4 (Figure 5) is challenging because students fail to consider that an integral is specified by both the region of integration as well as the integrand function, which in this case is $f(x, y, z) = z$. They vote that the given triple integral is equal to (a) the volume of a cube of side 1, because the limits of integration do indeed specify this region. However, because the integrand is not simply 1, this integral has a different meaning. Table 5 shows that we had four sections where a majority voted incorrectly for (a) but two sections where a majority voted correctly for (d), and another where the class was equally divided between three options. In the post-vote discussions, we found that students rarely computed the actual value of the integral, and were sometimes surprised when it was pointed out that knowing this value could be relevant to the question.

Many students struggle with the idea that there are an infinite number of ways to parameterize any curve, and find it difficult to recognize different parameterizations for the same curve. Question 5 (Figure 6) asks students to identify which of the given options is not a parameterization of the the entire curve $y = x^3$. Most students work through this in a superficial way, simply looking for the option which appears most different from the others, thus choosing (d), because this is the only option which includes coefficients other than one. To get students to consider the problem in more depth after a vote, it can be helpful to explicitly prompt students to try specific values for the parameter *t*, asking for the points that result, and whether or not these point are on the curve $y = x^3$.

Table 6 shows that this question prompted five of the seven sections to vote for the same incorrect answer, leaving section 4 where a majority of students voted correctly, and section 7 which was equally divided between (b) and (d). In section 4, the question was asked after students had worked through a worksheet introducing basic parameterizations, thus the question simply confirmed what the students already knew, rather than providing an opportunity to surprise the students and to produce a classroom environment in which the students were ready to learn.

To really understand what an integral means, students should be able to distinguish when a nontrivial integration is required, and when a calculation can be done with ordinary multiplication. Thus Question 6 (Figure 7) asks them which flux calculation requires more than ordinary multiplication. The most common error is to select (d), which includes a slightly more complicated vector field, involving r^2. However, the real issue is not how complicated the vector field is, but whether the component producing flux through the surface is a constant throughout that surface, or whether it varies, and thus requires a nontrivial integral. In option (d) it is the $\hat{r}$ component which produces a flux out of the sides of the cylinder, and the magnitude of this is given by r^2. But $r = r^2 = 1$ on the sides of the cylinder, and so ordinary multiplication will suffice to compute the flux.

Table 7 shows that this is a question which provoked very different results from the seven sections which voted it, even though only two instructors taught all of these sections. Indeed, sections 6 and 7 voted very differently, yet they were taught by the same instructor. Although the question is not reliable at producing a majority voting for this incorrect option, the question may be particularly valuable in providing feedback to the instructor, serving as a checkpoint to indicate whether this is a topic that students understand.

On three of these six questions we have only votes from classes taught by the original two instructors in the study, while the other three (Questions 1, 2, and 3) have votes from, the instructors brought in during the final year, teaching sections 10 and 11. We do not find any clear difference between the results from the various instructors. Section 10's vote on Question 2 is very consistent with the previous votes, and section 10's and 11's votes on Question 1 are similarly consistent. Question 3's common error was successfully navigated by section 10, but also by section 5. As a whole this suggests that the voting results studied here are not strongly dependent on the two instructors who began the study, but probably have a more general validity.

5 DISCUSSION AND CONCLUSIONS

We have found that in guiding a post-vote classroom discussion, these common-error questions allow the instructor to be particularly Socratic, calling on the students to explain and defend the options for which they voted, without offering assessment of their reasoning until after most of the class has figured it out for themselves. This discussion-leading technique is effective in this case because students are typically unafraid to explain and defend an opinion which is supported by the majority of the class, while at the same time there are usually students in the minority who are equally confident because they have recognized the error made by their peers.

One discussion-leading strategy is to first call on a student in the majority to explain their reasoning, and then to call on a student from the minority who voted correctly. If the voting is conducted anonymously with clickers, then this strategy would require calling on volunteers who have voted for specific answers. Getting students in the majority to speak up is rarely a problem, and often a student who voted correctly will volunteer to speak up with little prompting. Then, without evaluating the correct explanation, it can be useful to ask the first student to react to the explanation of the correct vote. At this point, the correct answer is usually clear and in this way, students learn their error, not from the instructor, but from a peer. This teaches students the value of listening to minority opinions in discussions. Occasionally, no one will vote correctly, or the students from the minority which voted correctly will be unable to articulate their reasoning sufficiently to persuade their peers. In these cases, it can be useful to simply announce to the class that the majority is incorrect, and to call for a second round of small group discussion and voting. In this way the instructor avoids explicitly telling the class what the right answer is, and instead encourages them to figure it out for themselves.

What are the characteristics of these rare questions which consistently provoke a majority of students to vote for a particular incorrect option? All six of these questions are well focused, directly dealing with only one key issue. Many questions simultaneously bring up a whole constellation of related issues, while the focus of these six questions makes them especially useful in this regard.

The common errors made in questions 3, 4, and 6 could all be avoided if the students worked through the problems in greater depth, evaluating the parameterizations at different points in question 3, sketching out the regions of integration in question 4, and actually computing the integral and other quantities in question 6. As a result, these particular questions serve to encourage students to actually work things out, rather than simply to trust their mathematical intuition. Perhaps one way of designing a common-error question is thus to think carefully about what answers might appear plausible if students do not perform the actual calculations.

The success of a question in producing a “time for telling” depends critically on when the question is asked, in addition to the content of the question itself. Question 5 clearly demonstrates that a particular question can produce very different responses depending on whether it is asked early or late in the lesson. As a result, a key characteristic of a successful common-error question may be that it is written in such a way that it can be posed early in the lesson, just after an idea has been introduced, when the students are more apt to make common errors. In drawing out these incorrect conceptions, we allow students the opportunity to remake their mental models in a memorable way

Ultimately, it would be almost impossible for anyone to consider our library of 317 classroom voting questions, and to anticipate that these six would be the ones to most effectively provoke common errors. Therefore, this study suggests that the recording and analysis of voting results is a worthwhile project, and that these past results can provide a useful guide to instructors who are making lesson plans and deciding which questions to use in an upcoming class period. We thus offer the "teacher's edition" of our voting question collection to any interested instructor who will send us an e-mail request.

Our experiences with using classroom voting in multivariable calculus have been very positive. We have found this teaching method to be effective at creating an active learning environment, where students regularly practice talking with their peers about important mathematical issues. However, the educational success of any particular vote rests on the quality of the question used, as well as the classroom context. When they are available, questions which provoke a majority of the class to make a common error and vote incorrectly can be remarkably effective teaching tools. These questions surprise students and pique their interest, creating memorable and dramatic teachable moments.

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Section	Instructor	Institution	Semester	Enrollment	Questions Voted
1	KC	Carroll	F06	22	112
2	MP	Carroll	F06	20	79
3	KC	Carroll	F07	24	153
4	MP	Carroll	F07	16	78
5	KC	Carroll	F08	23	142
6	KC	Carroll	F08	18	126
7	KC	Carroll	F09	21	155
8	MP	Carroll	F09	22	48
9	MP	Carroll	F10	18	33
10	HZ	Carroll	F10	24	81
11	AS	Hood	F10	21	31

Table 1: The eleven sections of multivariable calculus from which we gathered data

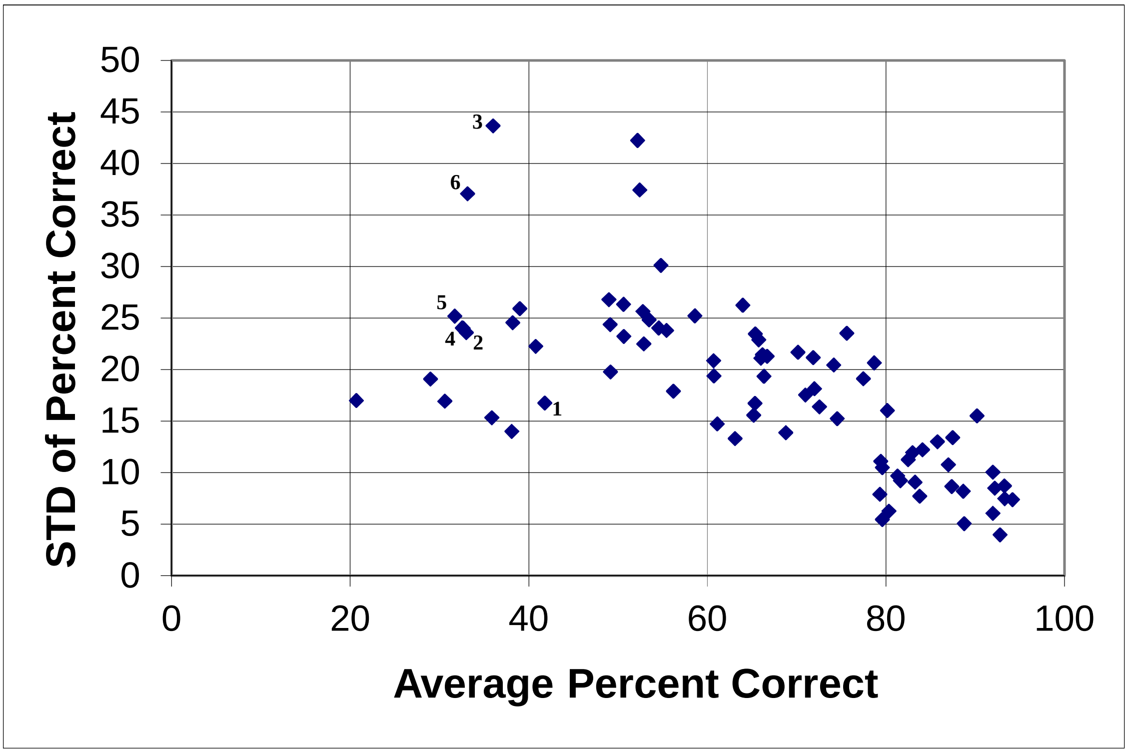

Figure 1: Here, for each of the 81 questions in the analysis, we plot out the standard deviation of the percentage of students voting correctly versus the average percentage of students voting correctly. Numbers indicate the six questions for which a majority of students voted for the same incorrect option.

The function $f(x, y)$ has gradient ∇f at the point (a, b). The vector ∇f is perpendicular to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$.

a) True

b) False

Figure 2, Question 1: The key idea of this question [10] is that the gradient of a function of two variables is a two dimensional vector, while the surface produced when we graph this function resides in three dimensions, thus in general this vector will not be perpendicular to the surface, and so the correct answer is (b) False.

	a)	**b)**	**Section**
	21%	**79%**	2
	68%	**32%**	5
	59%	**41%**	9
	76%	**24%**	10
	72%	**28%**	11
Avg.	*59%*	**41%**	
STD	*22%*	**22%**	

Table 2: Here we present the voting results from Question 1, with the correct answer indicated by the bold column. We see that in four of the five sections, a majority voted for the same incorrect response, as indicated with italics.

A small business has \$300,000 worth of equipment and 100 workers. The total monthly production, P (in thousands of dollars), is a function of the total value of the equipment, V (in thousands of dollars), and the total number of workers, N. The differential of P is given by $dP = 4.9\,dN + 0.5\,dV$. If the business decides to lay off 3 workers and buy additional equipment worth \$20,000, then

a) Monthly production increases.

b) Monthly production decreases.

c) Monthly production stays the same.

Figure 3, Question 2: This classroom voting question [10] calls for an application of the differential. The correct answer is (b), that monthly production decreases, because $dN = -3$ and $dV = +20$, so that $dP = -4.7$.

	a)	**b)**	c)	**Section**
	86%	**14%**	0%	1
	18%	**82%**	0%	3
	61%	**35%**	4%	5
	87%	**13%**	0%	6
	75%	**25%**	0%	7
	57%	**38%**	5%	8
	76%	**24%**	0%	10
Avg.	*66%*	**33%**	1%	
STD	*24%*	**24%**	2%	

Table 3: Here we present the voting results from Question 2, with the correct answer indicated by the bold column. We see that in six of the seven sections, a majority voted for the same incorrect response, as indicated with italics.

Which of the following integrals is equal to $\int_0^3 \int_0^{4x} f(x,y)\,dy\,dx$?

a) $\int_0^{4x} \int_0^3 f(x,y)\,dx\,dy$

b) $\int_0^{12} \int_{y/4}^3 f(x,y)\,dx\,dy$

c) $\int_0^{12} \int_3^{y/4} f(x,y)\,dx\,dy$

d) $\int_0^{12} \int_0^{y/4} f(x,y)\,dx\,dy$

e) $\int_0^3 \int_0^{4x} f(x,y)\,dx\,dy$

Figure 4, Question 3: Changing the order of integration usually requires an understanding of the geometrical region being described by the limits of integration. Here [10] this region has $y < 4x$, and so the correct answer is (b).

	a)	**b)**	**c)**	***d)***	**e)**	**Section**
	0%	**14%**	0%	*86%*	0%	3
	0%	**19%**	0%	*75%*	6%	4
	4%	**92%**	4%	*0%*	0%	5
	0%	**0%**	0%	*100%*	0%	6
	15%	**0%**	0%	*85%*	0%	7
	0%	**91%**	0%	*4%*	4%	10
Avg.	3%	**36%**	1%	*58%*	2%	
STD	6%	**44%**	2%	*44%*	3%	

Table 4: Here we present the voting results from Question 3, with the correct answer indicated by the bold column, and the most popular incorrect response indicated with italics.

What does the integral $\int_0^1 \int_0^1 \int_0^1 z\, dz\, dy\, dx$ represent?

a) The volume of a cube of side 1

b) The volume of a sphere of radius 1

c) The area of a square of side 1

d) None of the above

Figure 5, Question 4: When we compute this triple integral [10], we find that it equals $\frac{1}{2}$, while (a) is equal to 1, (b) is equal to $\frac{4}{3}$, and (c) is equal to 1, thus (d) is the correct answer.

	a)	b)	c)	**d)**	**Section**
	18%	0%	14%	**68%**	1
	68%	0%	9%	**23%**	3
	73%	7%	0%	**20%**	4
	37%	0%	0%	**63%**	5
	33%	0%	33%	**33%**	6
	86%	0%	9%	**5%**	7
	84%	0%	0%	**16%**	8
Avg.	*57%*	1%	9%	**33%**	
STD	*27%*	3%	12%	**24%**	

Table 5: Here we present the voting results from Question 4, with the correct answer indicated by the bold column, and the most popular incorrect response indicated with italics.

Which of the following is not a parameterization of the entire curve $y = x^3$?

a) $x(t) = t; y(t) = t^3$

b) $x(t) = t^2; y(t) = t^6$

c) $x(t) = t^3; y(t) = t^9$

d) $x(t) = 2t; y(t) = 8t^3$

Figure 6, Question 5: Here [11], the correct answer is (b), because these even powers return only positive values, and thus cannot produce the negative portion of the curve.

	a)	**b)**	**c)**	***d)***	**Section**
	22%	**9%**	9%	*61%*	1
	0%	**10%**	10%	*80%*	2
	0%	**14%**	8%	*86%*	3
	0%	**79%**	14%	*7%*	4
	0%	**36%**	4%	*60%*	5
	0%	**27%**	0%	*73%*	6
	0%	**47%**	6%	*47%*	7
Avg.	3%	**32%**	7%	*59%*	
STD	8%	**25%**	4%	*27%*	

Table 6: Here we present the voting results from Question 5, with the correct answer indicated by the bold column, and the most popular incorrect response indicated with italics.

All but one of the flux calculations below can be done with just multiplication, but one requires an integral. Which one?

a) $\vec{F} = 3\hat{\rho} + 2\hat{\phi}$ through a sphere of radius 4.

b) $\vec{F} = \rho\hat{\rho} + \theta\hat{\phi}$ through a sphere of radius 3.

c) $\vec{F} = r\hat{r} + r\hat{z}$ through a disk of radius 2, centered on the *z* axis, in the *z* = 2 plane.

d) $\vec{F} = r^2\hat{r} + z\hat{\theta}$ through a cylinder of radius 1, between *z* = 1 and *z* = 3.

Figure 7, Question 6: If an integrand is constant, then the integration can be computed with ordinary multiplication. Here, option (c) requires more than ordinary multiplication, because the $\hat{z}$ component of the vector field produces a flux through the disk, and the magnitude of this component is given by *r*, which varies from zero to two within the disk.

	a)	**b)**	**c)**	***d)***	**Section**
	0%	6%	**6%**	*88%*	1
	0%	30%	**9%**	*53%*	3
	0%	0%	**0%**	*100%*	4
	0%	0%	**54%**	*46%*	5
	0%	7%	**93%**	*0%*	6
	0%	5%	**5%**	*90%*	7
	0%	5%	**65%**	*30%*	8
Avg.	0%	8%	**33%**	*58%*	
STD	0%	10%	**37%**	*37%*	

Table 7: Here we present the voting results from Question 6, with the correct answer indicated by the bold column, and the most popular incorrect response indicated with italics.