Project: Artillery

The goal of artillery is to fire a shell (e.g. a cannon ball) so that it lands on a specific target. If we ignore the effects of air resistance, the differential equations describing its acceleration are very simple:

$$\frac{dv_x}{dt} = 0 \text{ and } \frac{dv_z}{dt} = -g$$

Here $g = 9.8$ m/s^2, and we can show that these equations result in a parabolic trajectory. Once we know this, we can easily calculate the initial speed v_0 and angle θ_0 above the horizontal necessary for the shell to reach the target, finding that the maximum range always results from an angle of $\theta_{0max} = 45°$.

However, the effects of air resistance are significant when the shell must travel a large distance, and if we modify these equations to include a simple model of air resistance, they become

$$\frac{dv_x}{dt} = -cv_x\sqrt{v_x^2 + v_z^2} \text{ and } \frac{dv_z}{dt} = -g - cv_z\sqrt{v_x^2 + v_z^2}.$$

Here the constant c depends on the shape and density of the shell and the density of air, but for this project assume that $c = 10^{-3}$ m^{-1}.

Now, imagine that you are living 200 years ago, acting as a consultant to an artillery officer who will be going into battle (perhaps against Napoleon). Although computers have not yet been invented, given a few hours or a few days to work, a person living in this time could project trajectories using numerical methods. Using this, you can try various initial speeds v_0 and angles θ_0 until you find a pair that reach any target. However, the artillery officer needs a faster and simpler method. He can do math, but performing hundreds or thousands of numerical calculations on the battlefield is simply not practical. Suppose that our artillery piece will be firing at a target that is a distance Δx away, and that Δx is approximately half a mile away – not exactly half a mile, but in that general neighborhood. Develop a simpler method of estimating v_0 and θ_0 with reasonable accuracy, given the exact range to the target Δx, and then test the effectiveness of your method by comparing it to the full numerical solution of these equations. (Use the midpoint method, and be sure to persuade me that your numerical solution is accurate enough.) If you have time, consider the effects of targets at different altitudes Δz, wind, and variations in the constant c. Hint: To get started, find v_0 and θ_0 to hit a target that is exactly half a mile away. Then find v_0 and θ_0 to hit a target that is slightly farther away than this.