Power Series Solutions

1. What is the solution to \( \frac{df}{dx} = f \)?
   (a) \( f(x) = e^x \)
   (b) \( f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots \)
   (c) Both of the above

2. Determine which of the following is a solution to the equation \( y'' - xy = 0 \) by taking derivatives and substituting them into the differential equation.
   (a) \( y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots \)
   (b) \( y = x + \frac{1}{4}x^4 + \frac{1}{14}x^7 + \frac{1}{105}x^{10} \cdots \)
   (c) \( y = 1 + \frac{1}{23}x^3 + \frac{1}{23\cdot5\cdot6}x^6 + \frac{1}{23\cdot5\cdot6\cdot8\cdot9}x^9 \cdots \)
   (d) \( y = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{2\cdot4\cdot6}x^6 + \cdots \)
   (e) None of the above

3. Write the series \( \sum_{n=2}^{\infty} (n+1)a_n x^n \) as an equivalent series whose first term corresponds to \( n = 0 \) rather than \( n = 2 \).
   (a) \( \sum_{n=0}^{\infty} (n+1)a_n x^n \)
   (b) \( \sum_{n=0}^{\infty} (n+3)a_{n+2} x^{n+2} \)
   (c) \( \sum_{n=0}^{\infty} (n-1)a_{n-2} x^{n-2} \)
   (d) \( \sum_{n=0}^{\infty} (n-1)a_{n+2} x^{n+2} \)

4. What is the second derivative of the function defined by the power series \( y(x) = \sum_{n=0}^{\infty} a_n x^n \)?
   (a) \( y''(x) = \sum_{n=0}^{\infty} n(n-1)a_{n-2} x^{n-2} \)
   (b) \( y''(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n \)
   (c) \( y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n \)
   (d) \( y''(x) = \sum_{n=0}^{\infty} (n-2)(n-3)a_{n-2} x^{n-4} \)
5. Find a power series in the form \( y(x) = \sum_{n=0}^{\infty} a_n x^n \) that is a solution for the differential equation \( y' = 3y \), by taking the derivative of this series, substituting both the series and its derivative into the differential equation, and simplifying the result.

(a) \( a_{n+1} = \frac{3}{n+1} a_n \)
(b) \( a_{n+1} = -\frac{3}{n+1} a_n \)
(c) \( a_{n+1} = \frac{3}{n-1} a_n \)
(d) \( a_{n+1} = -\frac{3}{n-1} a_n \)

6. Which of the following power series is a solution to the differential equation \( y' = -2y \)?

(a) \( y(x) = 2 - 2x + 2x^2 - \frac{4}{3} x^3 + \frac{2}{5} x^4 + \cdots \)
(b) \( y(x) = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \frac{2}{5} x^4 + \cdots \)
(c) \( y(x) = -2 - x - \frac{2}{5} x^2 - \frac{1}{2} x^3 + \cdots \)
(d) \( y(x) = 2 - 4x + 4x^2 - \frac{8}{3} x^3 + \frac{4}{3} x^4 + \cdots \)
(e) \( y(x) = 1 - 2x - 2x^2 - \frac{4}{3} x^3 - \frac{2}{5} x^4 + \cdots \)

7. Find a power series in the form \( y(x) = \sum_{n=0}^{\infty} a_n x^n \) that is a solution for the differential equation \( y'' + 4y = 0 \), by taking the second derivative of this series, substituting both the series and its second derivative into the differential equation, and finding a difference equation for \( a_n \).

(a) \( a_{n+2} = \frac{-4}{(n+1)(n+2)} a_n \)
(b) \( a_{n+2} = \frac{-4}{(n-1)(n-2)} a_n \)
(c) \( a_{n+2} = \frac{-4}{n(n+1)} a_n \)
(d) \( a_{n+2} = \frac{-4}{n(n-1)} a_n \)

8. Which of the following is a solution to \((x-2)y'' - y = 0\)?

(a) \( x + \frac{1}{7} x^2 + \frac{1}{12} x^3 + \frac{1}{144} x^4 + \cdots \)
(b) \((x-2) + \frac{1}{7} (x-2)^2 + \frac{1}{12} (x-2)^3 + \frac{1}{144} (x-2)^4 + \cdots \)
(c) \((x + 2) + \frac{1}{7} (x + 2)^2 + \frac{1}{12} (x + 2)^3 + \frac{1}{144} (x + 2)^4 + \cdots \)
(d) \(-144 \cdot (x-2) - 72(x-2)^2 - 12(x-2)^3 - (x-2)^4 + \cdots \)
(e) None of the above
(f) More than one of the above
9. Find a power series in the form \( y(x) = \sum_{n=0}^{\infty} a_n (x - 1)^n \) that is a solution for the differential equation \( y'' - xy = 0 \). Hint: After substituting in the series for \( y \) and \( y'' \) it may be useful to write \( x = 1 + (x - 1) \).

(a) \( a_{n+2} = \frac{a_{n+1} + a_n}{(n+1)(n+2)} \)

(b) \( a_{n+2} = \frac{a_n + a_{n-1}}{(n)(n+1)} \)

(c) \( a_{n+2} = \frac{a_n + a_{n+1}}{(n+1)(n+2)} \)

(d) \( a_{n+2} = \frac{a_n + a_{n+1}}{(n)(n+1)} \)